A Generalized Coherence Framework for Quantifying Input Contributions in Multi-Input Systems with Correlated or Uncorrelated Inputs

Nolan H Howes[a], Matthew S Allen[a], Dario Farina[b], Steven K Charles[a,c]

Affiliations[a]Mechanical Engineering, Brigham Young University; Provo, Utah

[b]Bioengineering, Imperial College London; London, United Kingdom

[c]Neuroscience, Brigham Young University; Provo, Utah

Corresponding Author: Nolan H Howes, nohowes@gmail.com

Abstract

In multi-input systems, it is often necessary to quantify the contribution of each input to an output. Such contribution analysis is frequently performed using a family of measures known collectively as coherence. However, when correlation is present between inputs, existing coherence measures do not accurately quantify the contribution of individual inputs, except in special cases. Here we propose an expanded coherence framework that enables contribution analysis in any multi-input system, regardless of input correlation.

We bridged the gap by defining three new coherence measures: component, excluded, and isolated coherence. Component coherence is an intermediate measure that decomposes measured output power into components attributable to inputs directly vs to interference between inputs. Strategically summing component coherence terms yields contributions from individual inputs, defined as either excluded coherence (the portion of the output that would be removed if a given input were excluded) or isolated coherence (the portion of the output that would remain if a given input were isolated). To demonstrate, we simulated a three-input mechanical system and compared both existing and novel coherence measures to the known contributions at varying levels of input correlation.

Only excluded and isolated coherence accurately estimated the true contributions at all levels of input correlation. Even when existing coherence measures accurately estimated true contributions, novel measures did the same, but with less random error. These new coherence measures represent a generalization of the existing framework; together with existing coherence measures, they enable accurate contribution analysis in multi-input systems regardless of input correlation.

Keywords

Coherence, contribution analysis, multiple-input, correlated inputs, optimum frequency response function, output decomposition, source identification

1 Introduction

Multiple-input systems are common in nature and engineering, and it is often desirable to understand how much each input contributes to an output. Examples of such systems include vibrations of various vehicle components generating observable acoustic noise for the driver [1], seismic ground motions causing rhythmic deformation of a structure [2], tremorogenic activity in multiple muscles producing tremor at the hand [3], and more [4-11]. In such situations, it is often of interest to identify how much each input contributes to the output to aid in targeted intervention that will maximally reduce the output response. In sum, estimating the contribution of individual inputs (frequently referred to as contribution analysis [1, 4, 5]) is a common problem.

In the simple case of uncorrelated inputs, contribution analysis can be accomplished in a straightforward manner using ordinary coherence [12]. Ordinary coherence is essentially a correlation performed in the frequency domain, returning a value between 0 and 1 that quantifies how linearly related two signals are as a function of frequency [13, 14]. If the two signals are (a) the input and output of a single-input system or (b) an input and output of a multi-input system with uncorrelated inputs, ordinary coherence between the two signals further represents the portion of the output that can be attributed to (i.e. caused by) the input [12]. In many multi-input systems, inputs are either naturally uncorrelated at a given frequency or, as is the case with most engineering systems, inputs can be controlled and made to be uncorrelated. In such systems, ordinary coherence is an obvious choice for contribution analysis. However, when inputs are mutually correlated, contribution analysis becomes significantly more challenging.

When inputs to a multiple-input system are correlated, special cases of contribution analysis can be performed using existing coherence measures. First, ordinary coherence provides an *upper bound* for the portion of the output that *could* be attributed to a given input [15]. Second, if correlation between inputs is thought to be the result of causal relationships between inputs, then partial coherence, which utilizes an iterative conditioning approach to redistribute the correlated portions of each input to their assumed sources, can be used to decompose the output into distinct contributions from each input [12]. Third, virtual coherence transforms the measured inputs into a set of "virtual" uncorrelated inputs and decomposes the output into contributions from each of these new inputs, but these contributions cannot be related back to the original inputs, so this method is primarily used simply to determine *how many* uncorrelated sources contribute significantly to the output [16]. Fourth, multiple coherence, which describes the frequency-dependent correlation between the full set of inputs and the output, can be used to determine the contribution of all inputs collectively [12].

Although useful in special cases, none of these existing measures is able to estimate the true contribution of a given input in the general case of correlated inputs. Existing coherence methods either ignore correlation between inputs (ordinary coherence), mathematically remove correlation between inputs, creating new inputs (partial/virtual coherence), or make no attempt to distinguish between correlated inputs (multiple coherence). Therefore, existing coherence measures cannot be used to perform contribution analysis for a multiple-input system with correlated inputs if (1) inputs cannot be controlled, (2) an upper limit on contribution is not sufficient, (3) contributions must be identified for individual inputs, and (4) contributions must be identified in terms of the original measured inputs.

Hence, the purpose of this work was to develop a new method of contribution analysis that accounts for correlation between the original, measured inputs and decomposes the output into contributions that can be directly attributed to each input. To this end, we have created a set of novel coherence measures as well as a generalized coherence framework to understand the relationships between existing and novel coherence measures. This framework allows for general contribution analysis of multiple-input systems with correlated inputs.

2 Analytical Methods

Throughout this paper, in keeping with the standard terminology of the field [12, 17], the term "input correlation" always refers to inter-input coherence. This clarification is necessary because correlation in the time domain does not imply coherence in the frequency domain, nor does a lack of correlation imply a lack of coherence [18].[1] When a robust quantification of input correlation is required, we use inter-input multiple coherence, which describes the coherence between a given input and the full set of other inputs (excluding the given input), i.e. the portion of the given input that can be recreated as a linear combination of all other inputs.[2]

2.1 Existing coherence measures and their interpretations in multiple-input systems with correlated inputs

To demonstrate how novel coherence measures relate to existing measures, we first describe existing coherence measures.

2.1.1 Ordinary Coherence

Ordinary coherence can be defined and interpreted in at least two ways [12]. First, ordinary coherence (γ_{xy}^2) between two signals (x and y) is most often calculated at a given frequency (f) as the magnitude-squared cross-power spectral density (CPSD) of the two signals (G_{xy}) normalized by the auto-power spectral densities (PSDs) of both signals (G_{xx} and G_{yy}) [12]:

$$\gamma_{xy}^2(f) = \frac{\left|G_{xy}(f)\right|^2}{G_{xx}(f)G_{yy}(f)} \tag{1}$$

Alternatively, and perhaps more intuitively for describing input-output relationships (with x representing the input signal and y representing the output signal), ordinary coherence can be expressed as a ratio of estimated to measured output power, where the estimated portion is found by passing the measured input through a single-input (SI) optimum frequency response function (FRF) denoted as L_{xy} (Figure 1A):

$$\gamma_{xy}^2(f) = \frac{\left|L_{xy}(f)\right|^2 G_{xx}(f)}{G_{yy}(f)} \tag{2}$$

For this formulation to equal the standard formulation, the estimated FRF must be the SI optimum FRF that minimizes the error between estimated and measured output power, often referred to as the H_1 estimator [12, 14]:

$$L_{xy}(f) = \frac{G_{xy}(f)}{G_{xx}(f)} \tag{3}$$

Under this formulation it is clear that ordinary coherence describes the portion of the measured output optimally[3] caused by the measured input, assuming that input and output are related via a SI linear system.

---

[1] Because coherence analysis is performed in the frequency domain, input correlation must be determined independently for all frequencies, such that a single set of inputs may be considered uncorrelated at some frequencies and correlated at others. For a set of inputs to be considered uncorrelated at a given frequency, ordinary coherence at that frequency must be zero between all input pairings.

[2] Inter-input multiple coherence is calculated in the same way as multiple coherence between inputs and an output (see section 2.1.4) but with the given input removed from the input set and treated as the output.

[3] Here and throughout the remainder of the paper, any reference to optimality refers to the definition of an optimum frequency response function, i.e. the linear system that minimizes error between the estimated and measured output power for a given set of measured input(s) and output. The optimum FRF is equivalent to the true system to the extent that the true system is linear and time-invariant, there are no unmeasured inputs that are correlated with measured inputs, output noise is uncorrelated with all inputs, and input noise is negligible [12].

Ordinary coherence can still be calculated for each input to a multiple-input (MI) system by applying Equation (1) or Equation (2) to each input, but if correlation is present between inputs, then ordinary coherence only provides an upper bound for the contribution of a given input to the output [15]. Ordinary coherence is always calculated assuming a SI relationship, even when a system is known to have multiple inputs. If an input is uncorrelated with all other inputs, then ordinary coherence between that input and the output describes a distinct portion of the output power that can only be attributed to the given input, and the ordinary coherence value can still be interpreted as the portion of the output caused by that input [12]. If all inputs are mutually uncorrelated, then ordinary coherence describes the distinct contribution of each input, and the set of ordinary coherence values will sum collectively to a value less than or equal to one, resulting in an ideal output decomposition[4] (Figure 1B) [12]. On the other hand, if correlation exists between the given input and other inputs, the SI optimum FRF does not take the possible contributions of other inputs into account; the error between power estimated from the given input and the measured output power is minimized, thereby maximizing ordinary coherence and providing an upper bound on the contribution from that input. In this case, the collective sum of ordinary coherence values is no longer constrained to be less than one, and the true output decomposition remains unknown (Figure 1C). In other words, in a MI system with correlated inputs, ordinary coherence describes the portion of the measured output optimally caused by the measured input *and* anything correlated with it.

2.1.2 Partial Coherence

Partial coherence addresses the issue of correlated inputs by assuming that correlation is the result of causal relationships between inputs [12]. If it is believed that correlation between inputs is present because causal relationships exist between inputs, then a conditioning approach can be employed to obtain a new set of uncorrelated inputs, where all original correlation has been attributed to its assumed source. This conditioning is accomplished by assigning an order to the inputs and progressively removing the portion of each subsequent input that is correlated with any previous input [12, 19]. Partial coherence ($\gamma^2_{iy\cdot(i-1)!}$) is then calculated as the ratio of estimated to measured output power, where the estimated portion is found by passing a conditioned input through its SI optimum FRF:

$$\gamma^2_{iy\cdot(i-1)!}(f) = \frac{\left|L_{iy\cdot(i-1)!}(f)\right|^2 G_{ii\cdot(i-1)!}(f)}{G_{yy}(f)} \tag{4}$$

Following the notation in [12], the " $\cdot$ " symbol is read as “conditioned on” or “uncorrelated with”, and $(i-1)!$ represents “all previous inputs”, such that $\gamma^2_{iy\cdot(i-1)!}$ indicates the coherence between input i and output y, where input i has been conditioned on all previous inputs. In general, partial coherence describes the portion of the measured output optimally caused by the measured input *and* anything correlated with it, *excluding* anything already correlated with previous inputs.

The results of a partial coherence analysis are highly dependent on the selected ordering of inputs, particularly if input correlation is high. Various methods have been proposed for selecting the ordering of inputs. Ideally, causality between inputs can be determined simply based on an understanding of the physical nature of the system itself [12]. Other proposed methods to estimate causal relationships include investigating phase relationships using the Hilbert transform [20] or signal lead/lag via cross-correlation in the time domain [12], or ordering inputs from largest to smallest ordinary coherence [12]. Importantly, if the input order does not

[4] We define an output decomposition as any breakdown, as a function of frequency, of the measured output PSD into portions that can be attributed to different sources (measured inputs, conditioned inputs, noise, etc.). Contribution analysis performed using coherence measures always results in an output decomposition. An *ideal* output decomposition is one whose individual components are distinct (i.e., each component can be attributed exclusively to a single source) and sum collectively to the multiple coherence (2.1.4). Partial coherence (2.1.2) and virtual coherence (2.1.3) both yield ideal output decompositions, but ordinary coherence (2.1.1) only provides an ideal output decomposition if inputs are uncorrelated.

correctly represent true causal relationships, then partial coherence results represent simply a mathematical convenience rather than the true output decomposition.

2.1.3 Virtual Coherence

Virtual coherence (sometimes called fractional coherence) uses eigenvalue decomposition or singular value decomposition to transform the set of correlated measured inputs into a new set of "virtual" uncorrelated inputs (for details, see [16, 21]). The output power can then be decomposed into distinct contributions from each of these virtual inputs using the same formulation as ordinary coherence. However, virtual inputs do not retain information regarding the original correlated inputs, so the resulting contributions cannot be interpreted in terms of the measured inputs [20]. As such, the primary use of virtual coherence is simply to identify the number of distinct, uncorrelated sources that contribute to the output [16, 20]. Because virtual coherence does not perform contribution analysis in terms of the original inputs, no further discussion of this measure is included here.

2.1.4 Multiple Coherence

Multiple coherence describes the portion of the output collectively caused by all inputs [12]. It can be calculated using MI optimum FRFs, which are the FRFs that minimize the error between estimated and measured output power while taking the contribution of all inputs into account simultaneously. These MI optimum FRFs (H_{iy}) are calculated for a q-input system as the MI extension of the H_1 estimator presented in Equation (3) [12, 14]:

$$\vec{H}_{xy}(f) = \begin{bmatrix} H_{1y}(f) \\ \vdots \\ H_{qy}(f) \end{bmatrix} = \begin{bmatrix} G_{11}(f) & \cdots & G_{1q}(f) \\ \vdots & \ddots & \vdots \\ G_{q1}(f) & \cdots & G_{qq}(f) \end{bmatrix}^{-1} \begin{bmatrix} G_{1y}(f) \\ \vdots \\ G_{qy}(f) \end{bmatrix} \quad (5)$$

Importantly, if inputs are uncorrelated, all off-diagonal cross-power terms are zero, such that each MI optimum FRF calculated using Equation (5) is equivalent to the corresponding SI optimum FRF (i.e., $H_{iy}(f) = L_{iy}(f)$ if inputs are uncorrelated) [12]. Whether or not inputs are correlated, multiple coherence ($\gamma^2_{y:x}$) can be calculated as the ratio of estimated to measured output power, where the estimated power is found by passing all inputs through their respective MI optimum FRFs and calculating

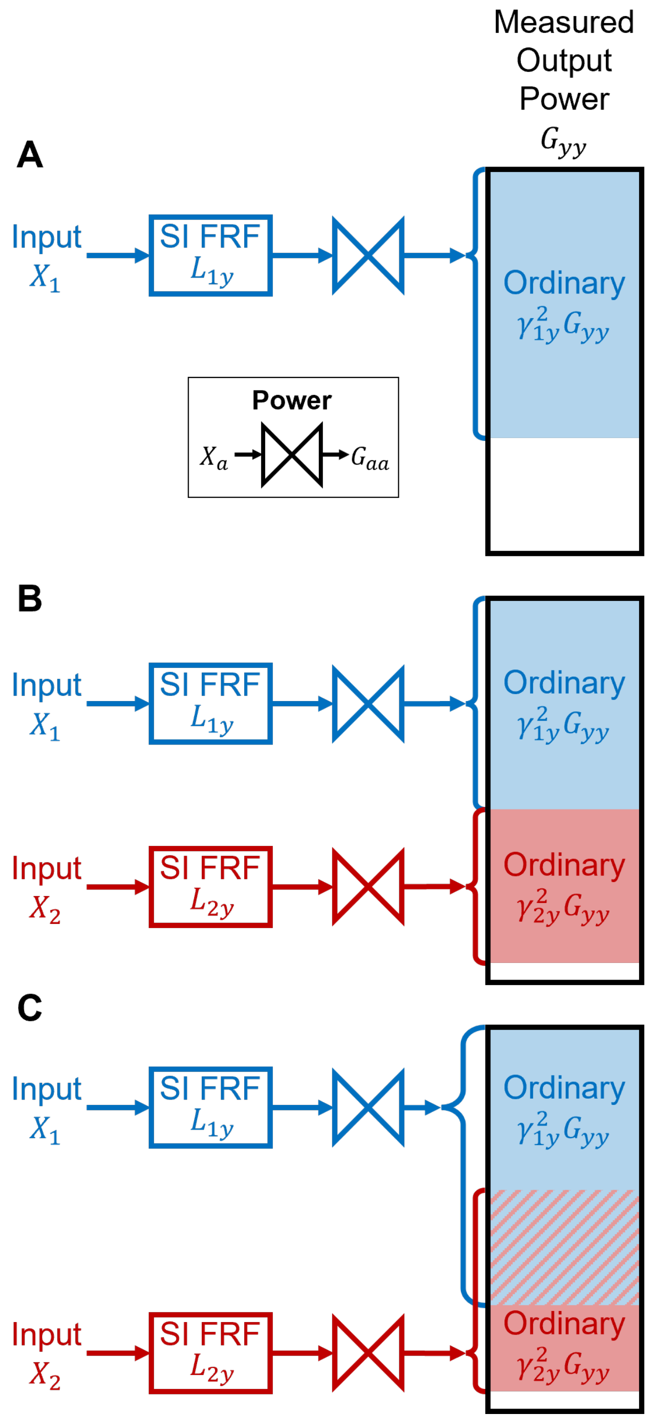

Figure 1. Graphical representation of ordinary coherence in systems with one input (A) or two inputs (B-C). A) In a single-input system, ordinary coherence describes the portion of the measured output power caused by the input via a single-input frequency response function (SI FRF). B) In a multiple-input system with uncorrelated inputs, the ordinary coherence associated with each input describes the distinct portion of the measured output power caused by each input. C) In a multiple-input system with correlated inputs, ordinary coherence cannot decompose output power into distinct contributions from each input, instead providing an upper bound on the contribution from each input. The area within each black-bordered box represents the total measured power of an output, and each colored shaded region represents the portion of the output power attributed to a given input by ordinary coherence. The sideways hourglass symbol represents the calculation of power for the given signal.

the power of the result (Figure 2A). Equivalently, to highlight the various components that make up this estimated power in terms of the original inputs, multiple coherence can be calculated as the sum of the power of the output of each FRF plus the sum of cross power between all possible pairings of these outputs (Figure 2B) [12]:

$$\gamma_{y:x}^2(f) = \frac{\vec{H}_{xy}^H(f)\begin{bmatrix} G_{11}(f) & \cdots & G_{1q}(f) \\ \vdots & \ddots & \vdots \\ G_{q1}(f) & \cdots & G_{qq}(f) \end{bmatrix}\vec{H}_{xy}(f)}{G_{yy}(f)} \tag{6}$$

$$= \frac{\sum_{i=1}^{q}\sum_{j=1}^{q} H_{iy}^*(f)H_{jy}(f)G_{ij}(f)}{G_{yy}(f)}$$

Alternatively, multiple coherence can also be calculated indirectly as the sum of all ordinary coherence terms for a set of uncorrelated inputs or as the sum of all partial or virtual coherence terms (even for correlated inputs) [12]. Multiple coherence describes the portion of the measured output optimally caused by the *full set* of measured inputs.

The primary drawback to multiple coherence is that it provides no insight into the contributions of individual inputs. The output decomposition accomplished by multiple coherence divides the output into two parts: the portion that could be linearly caused by the measured inputs and the portion that cannot [12]. As such, it is a useful measure in identifying system nonlinearities or noise [17] and provides an upper bound on all other existing coherence measures, but these are largely the extent of its applications.

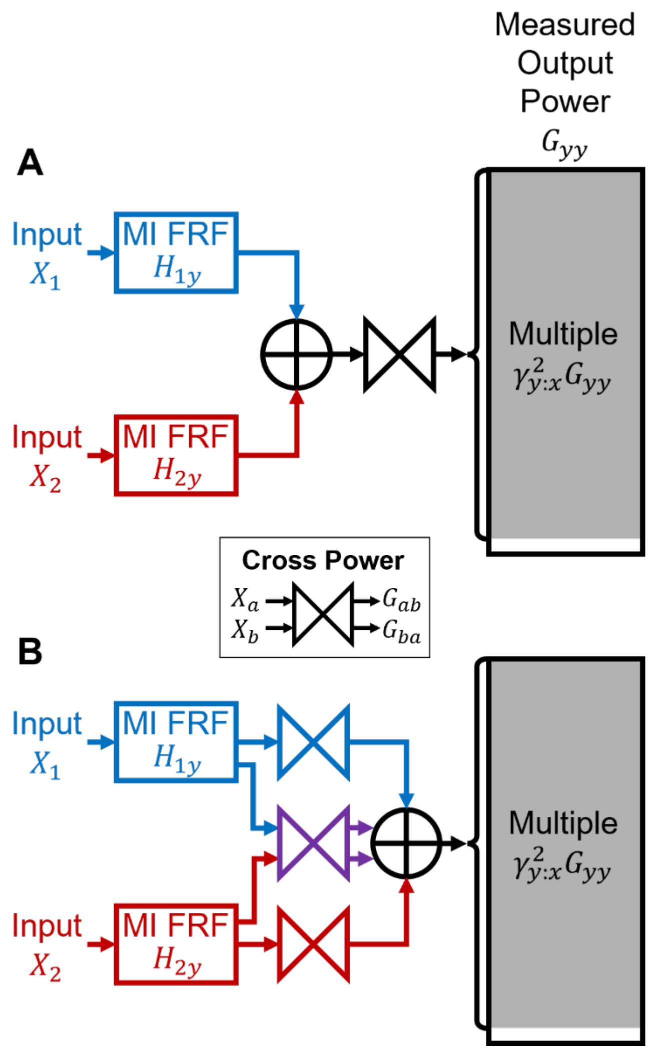

Figure 2. Graphical representation of multiple coherence in a system with two inputs. Multiple coherence describes the portion of the output power caused by all inputs collectively. It is calculated by passing each input through its respective multiple-input frequency response function (MI FRF), followed by either (A) summing the outputs and then calculating the power of the resulting sum, or (B) calculating the power of each output and the cross power between them and then summing all terms. Thus, A and B are equivalent representations. The power symbol (sideways hourglass, see Figure 1) with two input signals represents the calculation of both cross power terms between the two signals.

2.2 Novel Coherence Measures

Given the limitations of existing coherence measures in systems with correlated inputs, we propose a set of new coherence measures that account for correlation between the original measured inputs and estimate the true contribution of each input. To provide an intuitive understanding, we first present a conceptual explanation of these novel coherence measures, including a graphical representation of the simple 2-input case (Figure 3). We then expand this explanation to the general q-input case, with mathematical derivation and definition of each coherence measure.

2.2.1 Conceptual explanation

In a 2-input system with correlated inputs, the output can be decomposed into distinct components using a novel coherence measure which we propose to call component coherence. In such a system, each input passes through an FRF to generate a component of the output; these components then combine to create the total output. However, because the inputs are correlated these output components are also correlated, so interference between them affects the power of the resulting output. As a result, the total output power can

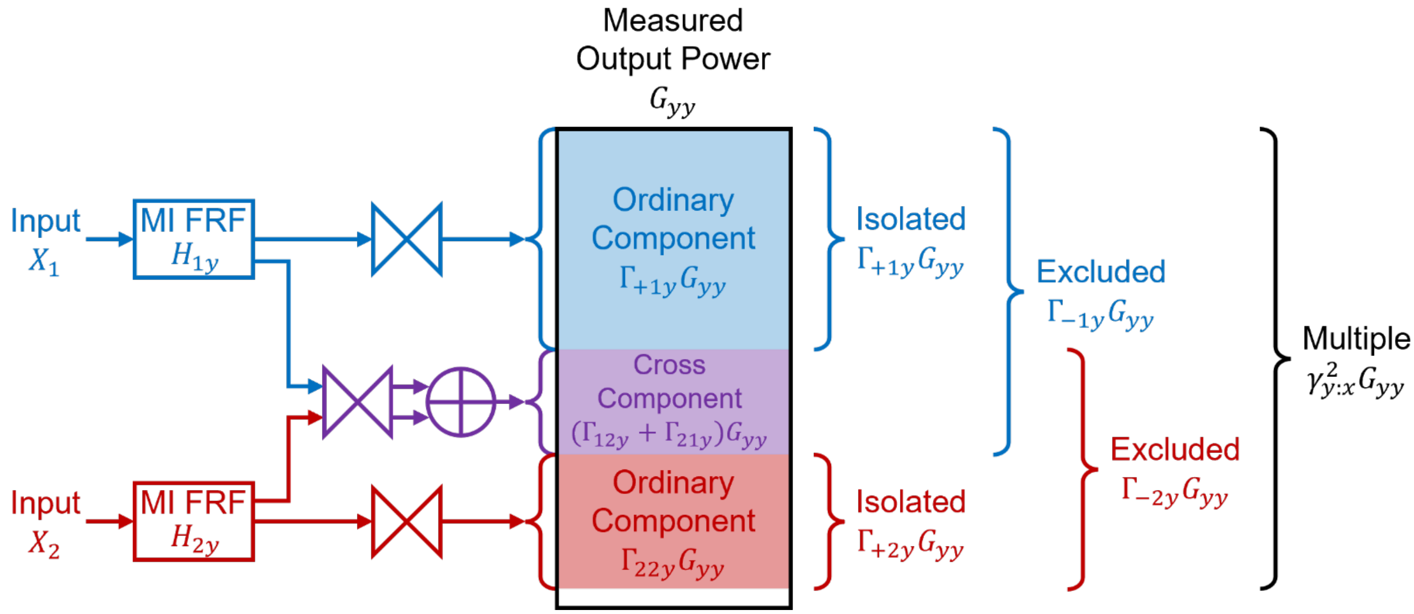

Figure 3. Graphical representation of novel coherence measures in a system with two inputs. Component coherence describes the portion of the output power that can be attributed to an input directly (ordinary component) or to interference between inputs (cross component) after the inputs pass through their respective multiple-input frequency response functions (MI FRFs). Isolated coherence describes the contribution that an input would make in isolation (i.e., in the absence of other inputs) and is equivalent to ordinary component coherence in this case. Excluded coherence describes the contribution that an input would make if it were excluded (i.e., the output power that would be removed if the input were removed) and is the sum of ordinary and cross component coherence in this case. Summing all component coherence terms gives multiple coherence.

be decomposed into three distinct portions: power directly from input 1, power directly from input 2, and power due to interference between them (Figure 3; blue, red, and purple shaded regions, respectively). Dividing the portions of power that come directly from a given input by the total measured output power defines a first type of component coherence which we call ordinary component coherence (Figure 3; blue and red shaded regions). Similarly, dividing the portion of power due to interference between the inputs by the total measured output power defines the second type of component coherence, called cross component coherence (Figure 3; purple shaded regions). Because interference can be constructive (power increases when signals are combined) or destructive (power decreases when signals are combined), the contribution due to interference, described by cross component coherence, may be positive or negative. Similarly, because destructive interference decreases the total power of the output, it is possible for the power contributed directly by a single input to be greater than the total resulting output power, indicated by an ordinary component coherence value greater than one.

Using component coherence, we can estimate the full contribution of each input using two additional novel coherence measures, which we will call excluded coherence and isolated coherence (Figure 3; brackets). Since a portion of the output power cannot be attributed exclusively to one of the inputs, but rather must be attributed to interference between them, we must more clearly define what is meant by contribution. A first possible definition of contribution, estimated using excluded coherence, is the portion of the total output power that would be removed if a given input were excluded. In the 2-input case, excluding a given input would remove the power it contributed directly to the output and the power it contributed through interference with the other input; as such, excluded coherence for a given input is calculated as the sum of the ordinary component coherence for that input and the cross component coherence between the inputs (importantly, the same cross component coherence portion is included in the excluded coherence for both inputs). Alternatively, another definition of contribution, estimated using isolated coherence, is the portion of the total output power that would remain if a given input were isolated (i.e. without all other inputs). If a single input were isolated, no interference would occur, so isolated coherence in this case is equivalent to ordinary component coherence (the more general case, in which isolated coherence differs from ordinary component coherence, is demonstrated in 2.2.2.3).

2.2.2 Mathematical derivation

Before presenting the details of these novel measures, we must highlight an important deviation from the standard concept of magnitude-squared coherence. All existing coherence measures are considered magnitude-squared coherence measures [12, 13, 22]. The concept of magnitude-squared coherence stems from the most common formulation of coherence as the magnitude-squared CPSD of the two signals, normalized by the PSD of both signals (Equation (1)) [13]. This definition of coherence is emphasized by the use of a lowercase gamma squared (γ^2) to denote all existing coherence measures. One of the most useful properties of magnitude-squared coherence is that these measures are bounded between zero and one [17]. However, for the purposes of estimating contributions in MI systems, magnitude-squared coherence is a limiting notion. In practice, as demonstrated in the following sections, it is not only useful, but also physically correct to consider contributions greater than one (where a given input contributes more power than that present in the measured output) and contributions less than zero (where the net effect of an input is to decrease power in the output). To avoid confusion, we present these novel coherence measures as contribution coherence measures (as opposed to magnitude-squared coherence measures) and denote them using a capital gamma without the square (Γ). Contribution coherence relies on the formulation of coherence used throughout this paper as a ratio of estimated output power to measured output power, where estimated output power may be larger or smaller than the measured power, positive or negative, and either real or complex. Note that, as demonstrated in 2.1, all existing coherence measures may also be considered contribution coherence measures when calculated for input-output relationships.

2.2.2.1 Component Coherence

Output power can be decomposed into components of power and cross power, associated with inputs and pairs of inputs respectively [12]. In the frequency domain, the value of the output of a q-input system at a given frequency can be calculated as the sum of each input ($X_i(f)$) passed through its FRF ($H_{iy}(f)$), plus a noise term ($N(f)$) that accounts for any deviation from the ideal linear model (including nonlinearities, unmeasured inputs, and measurement noise) [12]:

$$Y(f) = \sum_{i=1}^{q} H_{iy}(f) X_i(f) + N(f) \tag{7}$$

To calculate the power of the output at each frequency and the corresponding output power decomposition, each side of Equation (7) is multiplied by its conjugate to yield the following, where dependence on f has been omitted to simplify notation:

$$Y^*Y = \left[\sum_{i=1}^{q} H_{iy}^* X_i^* + N^*\right]\left[\sum_{j=1}^{q} H_{jy} X_j + N\right] = \sum_{i=1}^{q}\sum_{j=1}^{q} H_{iy}^* H_{jy} X_i^* X_j + \sum_{i=1}^{q} H_{iy}^* X_i^* N + \sum_{j=1}^{q} H_{jy} X_j N^* + N^* N \tag{8}$$

At this point, each instance of a conjugate signal multiplied by another signal can be replaced with the corresponding PSD or CPSD to result in a decomposition of the output PSD in terms of PSDs and CPSDs of and between inputs and the noise term (dependence on f is again omitted):

$$G_{yy} = \sum_{i=1}^{q}\sum_{j=1}^{q} H_{iy}^* H_{jy} G_{ij} + \sum_{i=1}^{q} H_{iy}^* G_{in} + \sum_{j=1}^{q} H_{jy} G_{nj} + G_{nn} \tag{9}$$

This result can be further simplified if the FRFs in the formulation represent optimum FRFs (Equation (5)), which guarantee that $N(f)$ will be uncorrelated with all inputs [12], making any cross-power term between $N(f)$ and an input go to zero and eliminating the two single summations from Equation (9):

$$G_{yy}(f) = \sum_{i=1}^{q}\sum_{j=1}^{q} H_{iy}^*(f) H_{jy}(f) G_{ij}(f) + G_{nn}(f) \tag{10}$$

Here, the double summation represents the portion of the measured output that can be linearly attributed to the full set of measured inputs, which if normalized by the total measured output power ($G_{yy}(f)$) provides the formulation for multiple coherence (Equation (6)):

$$\gamma_{y:x}^2(f) = \frac{\sum_{i=1}^{q}\sum_{j=1}^{q} H_{iy}^*(f)H_{jy}(f)G_{ij}(f)}{G_{yy}(f)} \tag{11}$$

An ideal output decomposition now becomes evident, consisting of q real power terms ($i = j$) that each represent power associated with a single input, and $q(q-1)$ complex cross-power terms ($i \neq j$) that each represent cross-power associated with interference between a pair of inputs; we will refer to these terms, each normalized by the output PSD, as ordinary component coherence and cross component coherence, respectively.

Ordinary component coherence (Γ_{iiy}) describes the portion of the measured output power contributed directly by an input. It is defined as the ratio of estimated to measured output power, where the estimation is made by passing the input through a MI optimum FRF:

$$\Gamma_{iiy}(f) = \frac{\left|H_{iy}(f)\right|^2 G_{ii}(f)}{G_{yy}(f)} \tag{12}$$

The value of this measure is always greater than zero (because power is positive) but has no upper bound. Ordinary component coherence describes the portion of the measured output optimally caused by *power directly* from an input.

Cross component coherence (Γ_{ijy}) can be used to describe the portion of the measured output power that can be attributed to interference between a pair of inputs. It is defined as the ratio of estimated output cross power to measured output power, where the estimation is made by passing a pair of inputs through their respective MI optimum FRFs and then calculating the cross power between them:

$$\Gamma_{ijy}(f) = \frac{H_{iy}^*(f)H_{jy}(f)G_{ij}(f)}{G_{yy}(f)} \tag{13}$$

Individual cross component coherence terms are complex, and therefore do not have a clear physical interpretation. However, when conjugate cross component coherence pairs (i.e., $\Gamma_{ijy}(f)$ and $\Gamma_{jiy}(f)$) are summed, the result is a real-valued term describing the portion of the measured output optimally caused by *interference* between a pair of inputs. As mentioned previously, the value of this sum may be positive or negative to describe constructive or destructive interference, respectively. The magnitude of this sum is always less than or equal to the sum of ordinary component coherence for the two inputs involved (constructive interference can increase power no more than double and destructive interference cannot decrease power below zero, see Supplemental Materials):

$$\left|\Gamma_{ijy}(f) + \Gamma_{jiy}(f)\right| \leq \Gamma_{iiy}(f) + \Gamma_{jjy}(f) \tag{14}$$

2.2.2.2 Excluded Coherence

By summing component coherence terms, we can define excluded coherence, which describes the portion of the measured output power that would be removed if a given input were excluded. For a given input, excluded coherence (Γ_{-iy}) is calculated as the sum of all component coherence terms that would go to zero if the input were removed; this includes the ordinary component coherence for the input, plus all cross component coherence terms involving the input:

$$\Gamma_{-iy}^2(f) = \sum_{j=1}^{q}\sum_{k=1}^{q} \Gamma_{jky}^2(f), \quad (j = i) \vee (k = i) \tag{15}$$

where the symbol ∨ represents the logical "inclusive or" operator, indicating that terms are summed if $j = i$ or $k = i$ or $j = k = i$. Excluded coherence describes the portion of the measured output power that would optimally be removed *if the input were excluded*.

Excluded coherence can also be defined for a subset of inputs. The definition remains unchanged but is now expressed in terms of any subset of inputs, $X_S = \{X_a, X_b, X_c, \dots\}$:

$$\Gamma^2_{-Sy}(f) = \sum_{j=1}^{q}\sum_{k=1}^{q} \Gamma^2_{jky}(f), \quad (j \in S) \vee (k \in S), \quad S = \{a, b, c, \dots\} \tag{16}$$

Excluded coherence for a subset of inputs represents the portion of the measured output power that would optimally be removed if the subset of inputs were removed.

There is no lower bound for excluded coherence, but excluded coherence can be no greater than one. Conceptually, excluded coherence can be no greater than one because removing a given input or subset of inputs can remove no more than 100% of the output power (i.e., the resulting output power must still be positive). In general, multiple coherence serves as the upper bound for excluded coherence (see Supplemental Materials). However, when excluded coherence is calculated for a single input, ordinary coherence serves as a more restrictive upper bound (see Supplemental Materials). A negative excluded coherence value indicates that output power would increase if the given input were removed (because of destructive interference).

2.2.2.3 Isolated Coherence

Like excluded coherence, isolated coherence is defined as a summation of component coherence terms, describing the portion of the output power that would remain if an input were isolated. Isolated coherence (Γ_{+iy}) for a single input is equivalent to ordinary component coherence, as this is the only component of output power that would remain if a single input were isolated:

$$\Gamma_{+iy}(f) = \Gamma_{iiy}(f) \tag{17}$$

For a subset of inputs, isolated coherence is calculated as the sum of all component coherence terms that would remain if that subset of inputs were isolated; this includes ordinary component coherence for each input plus all cross component coherence terms between inputs within the subset:

$$\Gamma^2_{+Sy}(f) = \sum_{j=1}^{q}\sum_{k=1}^{q} \Gamma^2_{jky}(f), \quad (j \in S) \wedge (k \in S), \quad S = \{a, b, c, \dots\} \tag{18}$$

Here, the symbol ∧ represents the logical "and" operator, indicating that terms are summed if $j \in S$ *and* $i \in S$. In all cases, isolated coherence is greater than zero (because isolated coherence describes power *remaining* when inputs are isolated, which is always positive), with no upper bound. Isolated coherence represents the portion of the measured output power that would optimally be contributed by the given input or set of inputs *if no other inputs were present*.

2.3 Generalized Coherence Framework

These novel coherence measures represent more than just an alternative toolset for analyzing correlated MI systems; they stand as a generalization of the existing coherence framework (Figure 4).

ALL coherence measures can be expressed as a ratio of estimated to measured output power (the most general definition of coherence):

$$Coherence = \frac{Estimated\ Output\ Power}{Measured\ Output\ Power} \tag{19}$$

*Table 1. Properties of existing and novel coherence measures (novel measures are marked with *). The novel coherence measures (ordinary component, cross component, excluded, and isolated coherence) bridge the gap between existing coherence measures (ordinary, partial, virtual, and multiple coherence). Frequency response function (FRF) type is either single input (SI) or multiple input (MI), calculated using Equation (3) or Equation (5), respectively.*

Coherence Type	Number of Inputs	Input Type	Power Type	FRF Type	Notation	Bounds
Ordinary	Single	Measured	Auto	SI	γ_{iy}^2	$0 \le \gamma_{iy}^2 \le 1$
Partial	Single	Conditioned	Auto	SI	$\gamma_{iy\cdot(i-1)!}^2$	$0 \le \gamma_{iy\cdot(i-1)!}^2 \le 1$
Virtual	Single	Virtual	Auto	SI	$\gamma_{i':y}^2$	$0 \le \gamma_{i':y}^2 \le 1$
Ordinary Component*	Single	Measured	Auto	MI	Γ_{iiy}	$0 \le \Gamma_{iiy} \le \infty$
Cross Component*	Pair	Measured	Cross	MI	Γ_{ijy}	$0 \le \lvert 2\mathrm{Re}(\Gamma_{ijy})\rvert \le \Gamma_{iiy} + \Gamma_{jjy}$
Excluded*	Single	Measured	Combined	MI	Γ_{-iy}	$-\infty \le \Gamma_{-iy} \le \gamma_{iy}^2$
Excluded*	Subset	Measured	Combined	MI	Γ_{-Sy}	$-\infty \le \Gamma_{-Sy} \le \gamma_{y:x}^2$
Isolated*	Single	Measured	Auto	MI	Γ_{+iy}	$0 \le \Gamma_{+iy} \le \infty$
Isolated*	Subset	Measured	Combined	MI	Γ_{+Sy}	$0 \le \Gamma_{+Sy} \le \infty$
Multiple	Full Set	Measured	Combined	MI	$\gamma_{y:x}^2$	$0 \le \gamma_{y:x}^2 \le 1$

This formulation highlights the fact that all coherence measures share the same fundamental interpretation, describing the portion of output power contributed by an input. Differences between measures arise in the types of inputs, types of FRFs, and types of power used for estimation (Table 1).

Component coherence is the general coherence measure from which all other coherence measures can be derived (Figure 4). Within component coherence, ordinary component coherence can be derived from cross component coherence by simply calculating cross power between a signal and itself (i.e., auto power). When conditioned inputs are substituted in place of measured inputs, ordinary component coherence becomes partial or virtual coherence, depending on which conditioning method is used. If measured inputs are uncorrelated to begin with, then ordinary component, partial, and virtual coherence are all equal to ordinary coherence. Summing component coherence terms (ordinary and cross) relevant to a single input or subset of inputs yields excluded or isolated coherence, depending on which summation scheme is used. Isolated coherence for a single input is equivalent to ordinary component coherence. Excluded coherence and isolated coherence are both equivalent to multiple coherence when calculated for the full set of inputs. In other words, multiple coherence

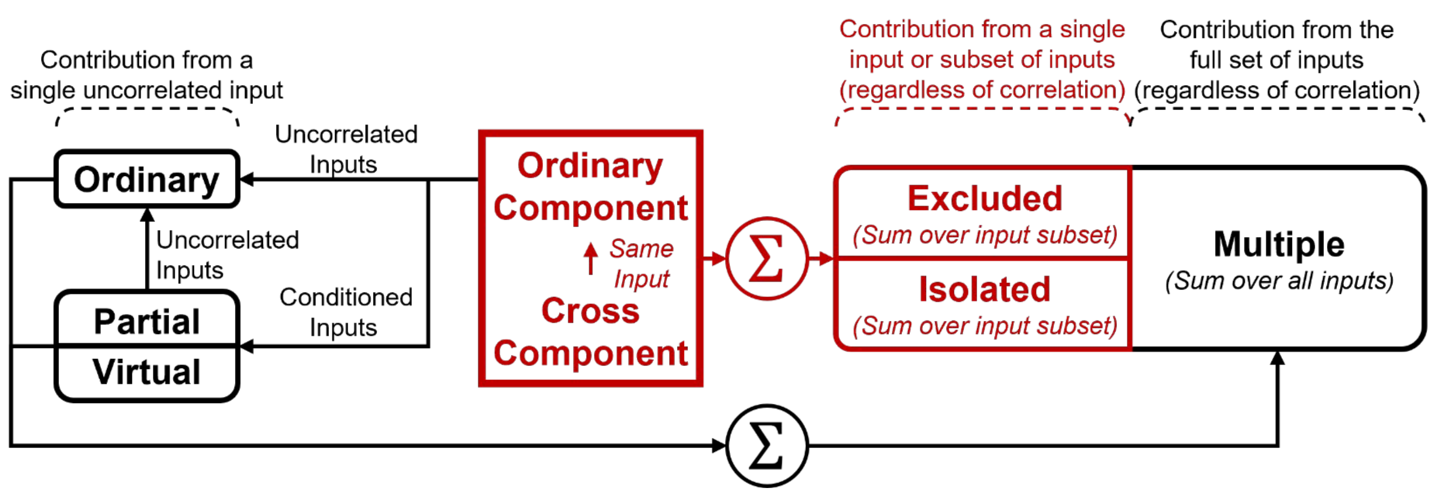

Figure 4. Generalized coherence framework, including relationships between existing coherence measures (black) and novel coherence measures (red). Cross component coherence can be used to derive all other coherence measures. Cross component coherence calculated between an input and itself gives ordinary component coherence, which further simplifies to ordinary coherence if inputs are naturally uncorrelated, or to partial or virtual coherence if inputs are mathematically uncorrelated via a conditioning process. Excluded and isolated coherence are obtained via strategic summation of component coherence terms, both of which simplify to multiple coherence when calculated for the full set of inputs. Multiple coherence can also be obtained by summing ordinary (if inputs are uncorrelated), partial, or virtual coherence across the full set of inputs.

is calculated as the full sum of all component coherence terms for a given system; similarly, multiple coherence can be calculated as the sum of all partial or virtual coherence terms for a conditioned system or as the sum of all ordinary coherence terms if the original measured inputs are uncorrelated. If measured inputs are uncorrelated, cross component coherence terms all go to zero and all other coherence measures for an individual input simplify to ordinary coherence (including isolated and excluded coherence).

Excluded and isolated coherence are the only measures that estimate all true contributions in any system, regardless of input correlation (Figure 4). Ordinary, partial, and virtual coherence can only describe the contribution of a single uncorrelated input; in the case of ordinary coherence, measured inputs must be uncorrelated to begin with, while partial and virtual coherence conditioning processes guarantee uncorrelated inputs (at the expense of interpretation). Multiple coherence can only describe the contribution of a full set of inputs, but this does hold regardless of input correlation. Excluded and isolated coherence bridge this gap by estimating the contribution of any input or subset of inputs in any system, regardless of input correlation.

3 Simulation Methods

3.1 System

To demonstrate the performance of these novel coherence measures and compare with existing measures, we simulated the behavior of a known mass-spring-damper model with three force inputs (Figure 5). The modeled system included four masses, three of which served as sites for force input, while the displacement of the fourth mass represented the system output. A parallel spring and damper were modeled between each possible pairing between masses and between each mass and ground. Parameter values for each mass, spring, and damper were randomly selected from a uniform distribution with limits selected based on parameter type to result in an underdamped system with natural frequencies below 50 Hz (masses: 0.1 to 1 kg, dampers: 3 to 30 N·s/m, springs: 1 to 10 kN/m). Parameter values used in the final model are shown in Table 2. The equations of motion for this system can be written in matrix form as:

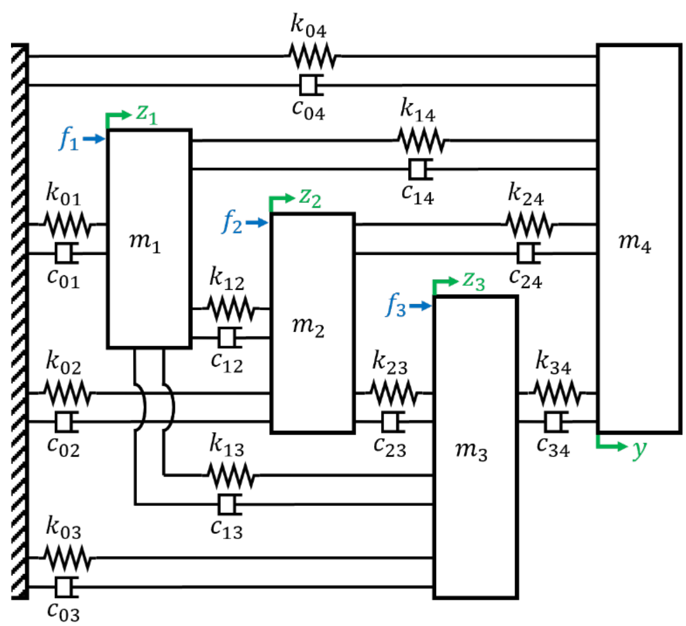

Figure 5. Modeled mass-spring-damper system used in simulation. Masses, damping coefficients, and stiffnesses are denoted by m, c, and k, respectively. Force inputs (f) are shown in blue and displacement outputs are shown in green. The displacement denoted by y represents the output of interest, whereas those denoted by z are included for modeling purposes only.

$$M\ddot{Z}(t) + C\dot{Z}(t) + KZ(t) = F(t) \tag{20}$$

where $F(t) = [f_1(t) \quad f_2(t) \quad f_3(t) \quad 0]^T$ and $Z(t) = [z_1(t) \quad z_2(t) \quad z_3(t) \quad y(t)]^T$ are the forces on the masses (inputs) and displacements of the masses (outputs), respectively. The mass, damping, and stiffness of the system are:

$$M = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix}, C = \begin{bmatrix} c_1 & -c_{12} & -c_{13} & -c_{14} \\ -c_{12} & c_2 & -c_{23} & -c_{24} \\ -c_{13} & -c_{23} & c_3 & -c_{34} \\ -c_{14} & -c_{24} & -c_{34} & c_4 \end{bmatrix}, K = \begin{bmatrix} k_1 & -k_{12} & -k_{13} & -k_{14} \\ -k_{12} & k_2 & -k_{23} & -k_{24} \\ -k_{13} & -k_{23} & k_3 & -k_{34} \\ -k_{14} & -k_{24} & -k_{34} & k_4 \end{bmatrix}$$

where $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} c_{01} + c_{12} + c_{13} + c_{14} \\ c_{02} + c_{12} + c_{23} + c_{24} \\ c_{03} + c_{13} + c_{23} + c_{34} \\ c_{04} + c_{14} + c_{24} + c_{34} \end{bmatrix}$ and $\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} k_{01} + k_{12} + k_{13} + k_{14} \\ k_{02} + k_{12} + k_{23} + k_{24} \\ k_{03} + k_{13} + k_{23} + k_{34} \\ k_{04} + k_{14} + k_{24} + k_{34} \end{bmatrix}$.

3.2 Inputs and output

To highlight the performance of novel and existing coherence measures under varying levels of input correlation (inter-input multiple coherence), we designed four different sets of the three simulated force inputs, each set having a different level of input correlation: minimal correlation (Case 1), moderate correlation (Case 2), high correlation (Case 3), and high correlation between inputs 1 and 2 but not 3 (Case 4). In addition, we wished to demonstrate the performance of novel and existing coherence measures under different levels of input-output multiple coherence, so we designed the four cases to have high input-output multiple coherence (representing favorable measurement conditions) over one frequency band (below ~35Hz) and low input-output multiple coherence (representing poor measurement conditions) over another frequency band (above ~35 Hz). In each of the four cases, the inputs and output were created as follows (Figure 6).

Table 2. Simulated system parameters. M, C, and K represent mass, damping, and stiffness, respectively.

$M\ [kg]$	$C\ [^{N\cdot s}/_{m}]$	$K\ [^{kN}/_{m}]$
$m_1 = 0.6$	$c_{01} = 21$	$k_{01} = 8.3$
$m_2 = 1.0$	$c_{02} = 24$	$k_{02} = 2.4$
$m_3 = 0.8$	$c_{03} = 13$	$k_{03} = 5.8$
$m_4 = 0.9$	$c_{04} = 20$	$k_{04} = 3.5$
	$c_{12} = 6$	$k_{12} = 5.0$
	$c_{13} = 7$	$k_{13} = 6.5$
	$c_{14} = 14$	$k_{14} = 9.8$
	$c_{23} = 6$	$k_{23} = 5.6$
	$c_{24} = 5$	$k_{24} = 2.0$
	$c_{34} = 8$	$k_{34} = 8.4$

3.2.1 Generation of correlated inputs

Four independent, white gaussian noise signals were generated using MATLAB's *randn* function, each 40 seconds in duration with a sampling frequency of 1000 Hz. Each signal was lowpass filtered using a 16th-order Butterworth filter with cutoff at 35 Hz so that the later addition of measurement noise (see below) would cause the 35-50 Hz band to be dominated by noise, resulting in low input-output multiple coherence in that band. The cutoff of 35 Hz was selected to ensure that high-quality signals were passed at the prominent natural frequencies of the system (below ~35 Hz) while still demonstrating the behavior of poor-quality signals at frequencies where inputs were not completely attenuated by the system (~35-50 Hz); the high filter order provided a steep cutoff to clearly distinguish these two bands (a 4th-order filter was also tested and results were unaffected).

In each of the four cases, these four independent, filtered signals were then combined using a mixing matrix, A, to generate the three inputs to the system with the desired level of input correlation. Each parameter a_i was selected as a number between zero and one describing the fraction of shared signal to be included in input i.

$$A = \begin{bmatrix} 1-a_1 & 0 & 0 & a_1 \\ 0 & 1-a_2 & 0 & a_2 \\ 0 & 0 & 1-a_3 & a_3 \end{bmatrix}$$

In each of the four cases, the correlation level was defined in terms of the desired level of mean inter-input multiple coherence below 35 Hz (the band of high-quality signal), and parameter values (a_i) were selected to achieve the desired values (Table 3). Minimal correlation (Case 1) was obtained when no mixing occurred (all $a_i = 0$), resulting in inter-input multiple coherence values of approximately 0.14. Moderate correlation (Case 2) was defined as inter-input multiple coherence of approximately 0.50 (all $a_i = 0.55$), and high correlation (Case 3) was defined as inter-input multiple coherence of approximately 0.90 (all $a_i =$

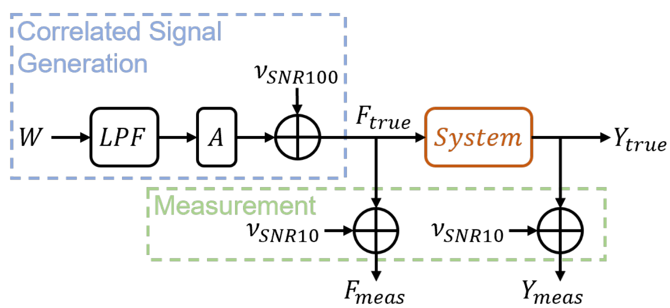

Figure 6. Simulation process. True correlated force input signals (F_{true}) were generated (blue box) as gaussian white noise signals (W) passed through a low-pass filter (LPF), mixed according to mixing matrix A to generate desired levels of correlation, and combined with noise (ν) at a signal-to-noise ratio (SNR) of 100 to effectively eliminate correlation above the filter cutoff frequency. The true force inputs were passed through the system to generate the true output (Y_{true}). Measurement conditions were simulated by adding noise to both the inputs and output at a SNR of 10, generating measured inputs (F_{meas}) and output (Y_{meas}).

Table 3. Levels of inter-input correlation used in simulations

Input Correlation Level	Mean Inter-Input Multiple Coherence (0-35 Hz)	Parameters
Case 1: Minimal Correlation	~0.14	$a_1 = a_2 = a_3 = 0$
Case 2: Moderate Correlation	~0.50	$a_1 = a_2 = a_3 = 0.55$
Case 3: High Correlation	~0.90	$a_1 = a_2 = a_3 = 0.79$
Case 4: High Correlation Between Inputs 1 & 2	Inputs 1 & 2: ~0.90, Input 3: 0.13	$a_1 = a_2 = 0.81,\ a_3 = 0$

0.79). Additionally, high correlation between inputs 1 and 2 but minimal correlation with input 3 (Case 4) was included ($a_1 = a_2 = 0.81, a_3 = 0$) to analyze the performance of coherence measures when correlation is present between only some of the inputs. To effectively eliminate input correlation above 35 Hz, we added additional noise to each input at a signal-to-noise ratio of 100.

3.2.2 True inputs and outputs

In each of the four cases, we used MATLAB's *lsim* function to find the true response (output signal, y) of the mass-spring-damper system to these three input signals (Figure 6).

3.2.3 Addition of measurement noise

Additional noise was added to the inputs and output to represent measurement noise, each resulting in a signal-to-noise ratio of 10 (Figure 6). To remove the effects of transient response, the first 10 seconds of each input and output signal were discarded, leaving only the final 30 seconds for further analysis.

3.3 Contributions

The following processing steps were performed for each of the four cases of input correlation.

3.3.1 True contributions

True contribution values were found by selectively excluding or isolating inputs and comparing the resulting outputs to the original output, using the true (noise-free) input and output signals. Specifically, the excluded contribution was found by excluding the selected input in simulation (setting it to zero), re-running $lsim$, and calculating the difference between the original output PSD ($G_{yy}(f)$) and the PSD of the output simulated with input i excluded ($G_{y_{-i}y_{-i}}(f)$), normalized by the original output PSD:

$$\frac{G_{yy}(f) - G_{y_{-i}y_{-i}}(f)}{G_{yy}(f)}$$

To find the isolated contribution of each input, we isolated the selected input in simulation (set all other inputs to zero), re-ran *lsim*, and calculated the PSD of the output simulated with input i isolated ($G_{y_{+i}y_{+i}}(f)$), normalized by the PSD of the original output:

$$\frac{G_{y_{+i}y_{+i}}(f)}{G_{yy}(f)}$$

True contributions were expressed as fractional contributions (i.e., normalized by the full output PSD) to allow for direct comparison with coherence measures. All PSDs were calculated using MATLAB's *cpsd* function with 18 windows and 50% overlap.

3.3.2 Coherence

All coherence values (excluding virtual coherence, which cannot be interpreted in terms of the original inputs) were calculated from the noisy inputs and output using the formulas presented in Section 2. Again, all PSDs and CPSDs were calculated using MATLAB's *cpsd* with 18 windows and 50% overlap. Multiple coherence was calculated for the full set of inputs (Equation (6)). Ordinary, excluded, and isolated coherence were calculated for each input individually (Equation (2), Equation (15), and Equation (17), respectively). Partial coherence was calculated for all three possible conditionings of each input (Equation (4)), e.g. input 3 conditioned on input 1, on input 2, and on inputs 1 & 2 (see [12] for details regarding the conditioning process).

3.3.3 Error

Finally, we calculated the mean error and the root-mean-square (RMS) error for each coherence measure (excluding multiple coherence—see below) relative to each true contribution (excluded, isolated) in the band from 0 to 35 Hz. Mean error was calculated as the estimated contribution (coherence) minus the true contribution, averaged across the frequency band. RMS error was calculated using MATLAB's *rmse* function. Mean error was used to quantify the bias error of a given coherence measure relative to each true contribution type. RMS error quantifies the total error of an estimator, capturing both bias error and random error [12]; as such, RMS error was used in cases when mean error indicated negligible bias error to quantify the random error of a given coherence measure.

For each of the four cases of input correlation (minimal, moderate, high, and high between only two inputs), we reported the average and standard error of the mean error and RMS error across all three inputs (f_1, f_2, f_3) for each coherence measure (ordinary, partial, excluded, and isolated) relative to both contribution types (true excluded contribution and true isolated contribution). In the case when only two inputs were highly correlated (Case 4), only those two inputs were included in calculating the average and standard error. Mean error and RMS error were not calculated for input-output multiple coherence because it does not allow contributions of individual inputs to be estimated.

4 Simulation Results

4.1 General results

As designed, the modeled system (Figure 5) was underdamped, with natural frequencies below 50 Hz (Figure 7A-C). In each of the four cases (of specified level of input correlation), we calculated the true contributions and all coherence values for each input (Figure 7D-O). As planned, input-output multiple coherence was high below 35 Hz and low above 35 Hz. When input-output multiple coherence was high (0-35 Hz), excluded coherence consistently estimated the excluded contribution and isolated coherence consistently estimated the isolated contribution, in every case. In contrast, ordinary and partial coherence estimated the true contribution only when input correlation was minimal (Figure 7D-F, also O). When multiple coherence was low (above 35 Hz), all coherence measures failed to estimate the true contributions.

In some cases, the true excluded contribution and excluded coherence took on negative values at certain frequencies (see particularly Figure 7H and K); in these cases, the output power at those frequencies increased when the given input was excluded, indicating that the given input had interfered negatively with the other inputs.

4.2 Mean error

In the 0 to 35 Hz band, excluded coherence was the only measure that estimated the excluded contribution with negligible mean error at all levels of input correlation, and isolated coherence was the only measure that estimated the isolated contribution with negligible mean error at all levels of input correlation (Figure 8A-B). Ordinary coherence, which generally provides an upper bound on contribution (see details above), resulted in positive mean error relative to both contribution types at all input correlation levels. The mean error associated with ordinary coherence was lowest when inputs were minimally correlated (Case 1) and greatest when inputs were highly correlated (Case 3). Partial coherence, which reassigns portions of contribution to other coherent inputs, generally resulted in negative mean error relative to both contribution types, with lowest error for minimally correlated inputs (Case 1) while conditioning on only a single input and greatest error for highly correlated inputs (Case 3) while conditioning on both other inputs. Partial coherence resulted in positive mean error only when the given input was conditioned on a single uncorrelated input, in which case conditioning has little effect and partial coherence is comparable to ordinary coherence.

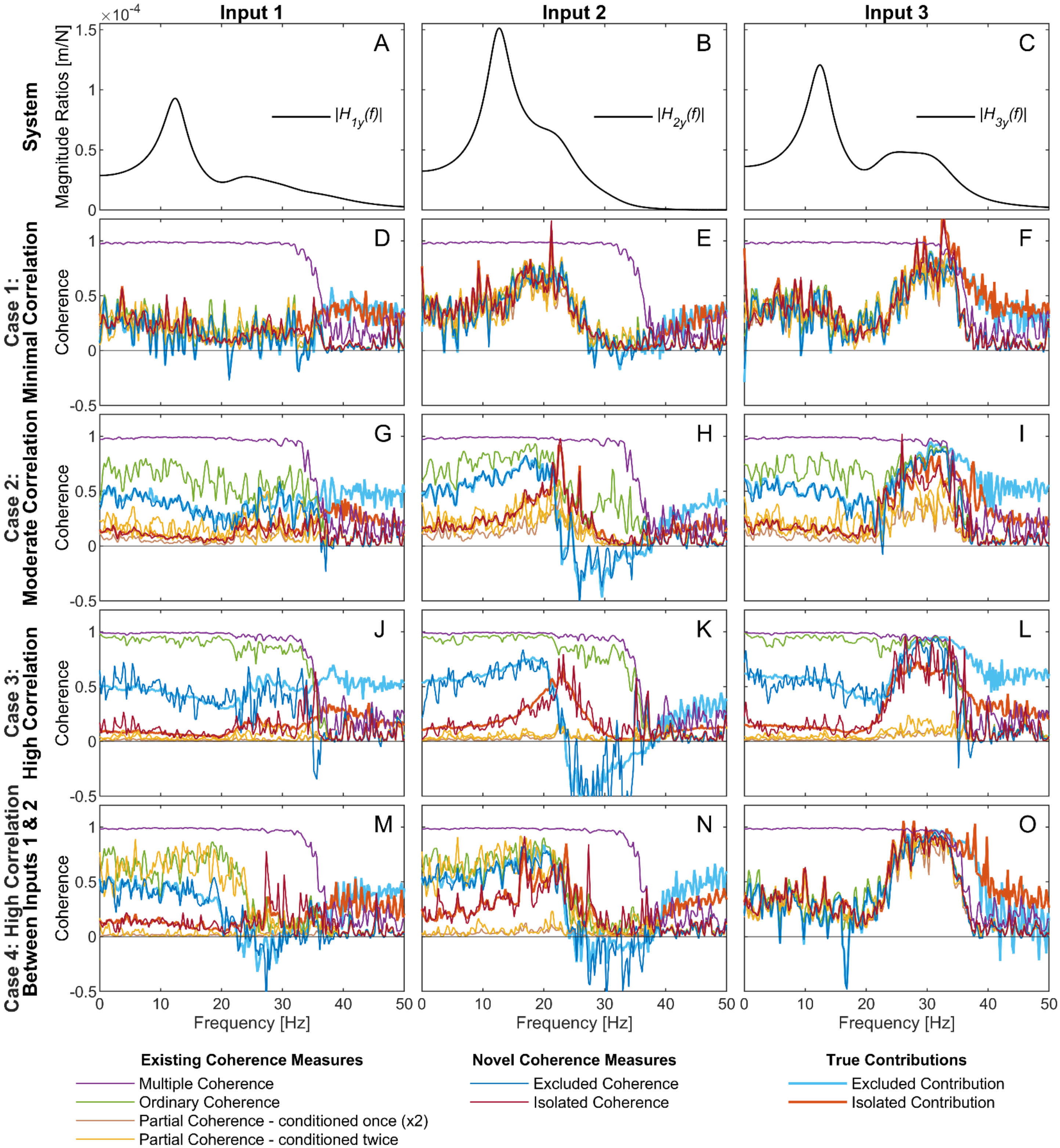

Figure 7. Simulation Results. A-C) Magnitude ratio of the frequency response function ($|H_{iy}(f)|$) between each input, i, and the output, y. D-O) Coherence vs. frequency plots for all coherence types between the specified input (column) and the output at different levels of input correlation (row). The coherence types are indicated in the legend at the bottom of the figure. Each coherence vs. frequency plot includes three lines for partial coherence: two thin yellow lines each representing partial coherence for the input conditioned on one of the two other inputs (e.g. input 1 conditioned on input 2, and input 1 conditioned on input 3), and one thin orange line representing partial coherence for the input conditioned on both other inputs (e.g. input 1 conditioned on both input 2 and input 3).

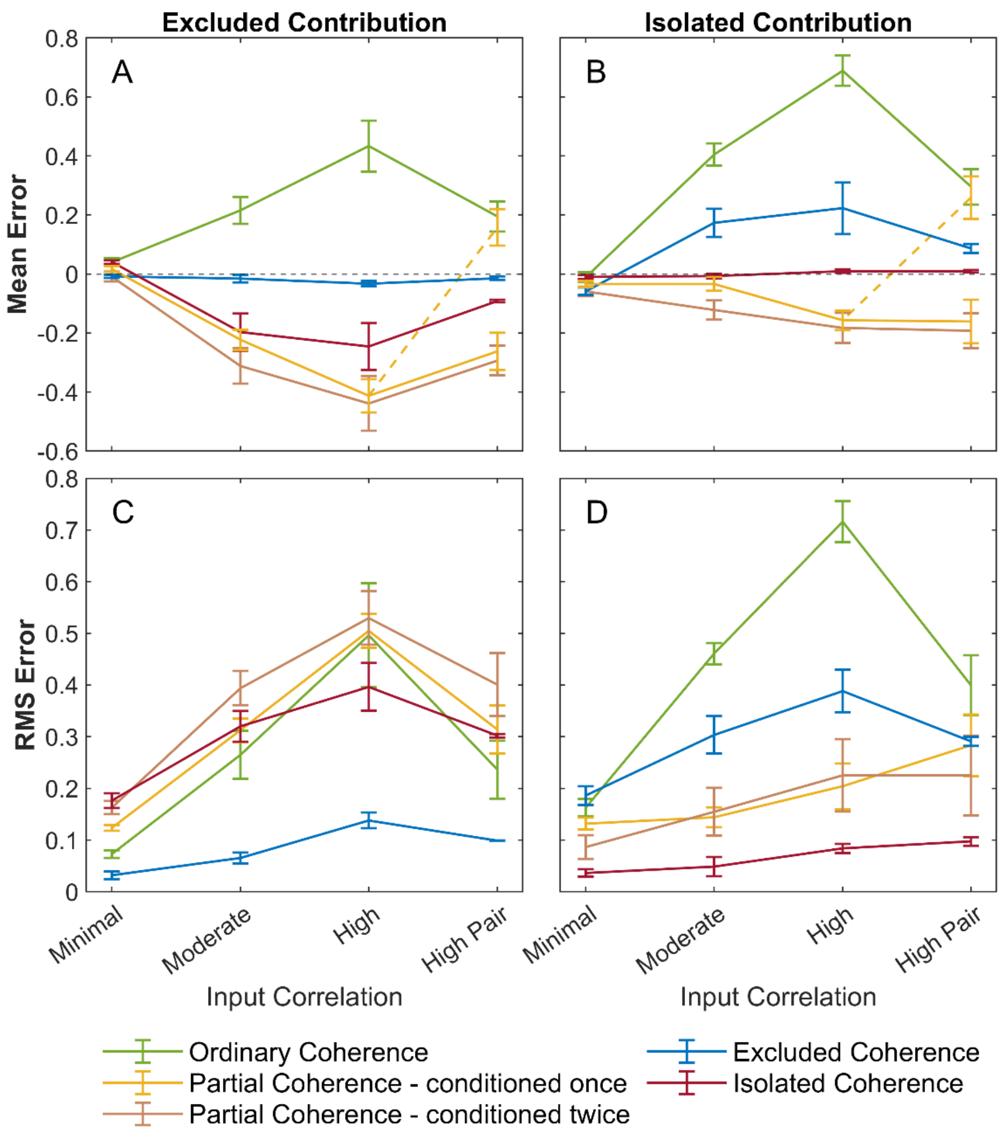

Figure 8: Simulated error between true contributions and coherence measures. Bars represent ±1 standard error. Each value represents the mean error (A-B) or RMS error (C-D) between the specified true contribution type (column) and the given coherence measure averaged across all three inputs for the indicated level of input correlation, except for the "High Pair" case. In this case (in which inputs 1 and 2 were correlated, but not input 3), input 3 was excluded in calculating the average error values; also, mean error was calculated separately for partial coherence conditioned only on the correlated input (solid line) and for partial coherence conditioned only on the uncorrelated input (dashed line).

4.3 Root-mean-square error

In the 0 to 35 Hz band, excluded coherence estimated the excluded contribution with the lowest RMS error, and isolated coherence estimated the isolated contribution with the lowest RMS error, both regardless of input correlation (Figure 8C-D). When bias is negligible (which is always the case for excluded and isolated coherence with respect to their corresponding contribution types, but is true for other coherence measures only when input correlation is minimal), the RMS error can be used as a quantification of the random error in the estimation [12]. RMS error between excluded coherence and the excluded contribution increased with increasing input correlation. Similarly, RMS error between isolated coherence and the isolated contribution increased with increasing input correlation. Even in Case 1, where input correlation was minimal (and the errors associated with existing coherence measures were smallest), the RMS error relative to the excluded contribution was still lowest for excluded coherence, and the RMS error relative to the isolated contribution was still lowest for isolated coherence.

5 Discussion

5.1 Interpretation of Results

The results of our simulations demonstrate four main principles regarding the estimation of input contribution using coherence measures.

(1) Multiple coherence defines the region where other coherence measures may accurately estimate the true contribution of each input (Figure 7). When multiple coherence was high, at least one coherence measure was able to consistently estimate each true contribution in every case. When multiple coherence was low, no linear relationship existed between the measured inputs and outputs, so the true contribution could not be estimated using coherence measures. Given that, in practice, multiple coherence may be less than unity due to any of several possible factors (e.g., noise, nonlinearities, unmeasured inputs), it is unhelpful to define a threshold (at which multiple coherence is too low) based on these simulations, in which low multiple coherence is due exclusively to measurement noise; thoroughly defining such a threshold in relation to all possible factors is beyond the scope of this paper. Where needed, a rough multiple-coherence threshold of 0.5 or greater may be useful, as suggested in [12].

(2) Excluded and isolated coherence are the only coherence measures that consistently estimate the true contributions when a given input is correlated with other inputs (Figure 7). In every case, excluded coherence consistently estimated the true excluded contribution with negligible bias (Figure 8A). Similarly, in every case, isolated coherence consistently estimated the true isolated contribution with negligible bias (Figure 8B). As an upper bound on the excluded contribution, ordinary coherence always overestimated the true excluded contribution. Ordinary coherence also, on average, overestimated the true isolated contribution, though not always; at some specific frequencies, the isolated contribution was greater than the ordinary coherence. Partial coherence conditioned on correlated inputs tended to underestimate the true contributions because it assigns too much contribution to other inputs (except when the true excluded contribution is negative). Partial coherence conditioned on an uncorrelated input was comparable to ordinary coherence (see Figure 7M-N).

(3) Though smaller than for existing coherence measures, random error (quantified as RMS error) in the estimates of excluded and isolated coherence for a given input increases with higher correlation between inputs (Figure 8C-D). When correlation between inputs increased, excluded and isolated coherence estimated the true contributions with increased random error. This increase was more pronounced from moderate to high input correlation than it was from minimal to moderate input correlation, suggesting a more than linear increase in random error with increasing input correlation. In the theoretical limiting case of perfectly correlated inputs, the input spectral density matrix (the q-by-q matrix in Equation (5)) becomes singular and cannot be inverted to estimate the system FRFs, so no estimate of excluded or isolated coherence can be made (essentially, infinite error).

Random error for excluded and isolated coherence also depends on the *number* of correlated inputs, even when total input correlation (quantified by inter-input multiple coherence) remains the same. For excluded coherence, the random error decreased when fewer inputs were correlated, even when the level of input correlation remained the same (Figure 8C). For isolated coherence, the random error increased slightly when fewer inputs were correlated, but this increase was not significant (Figure 8D). Future work is needed to investigate this principle more thoroughly.

(4) When a given input is minimally correlated with other inputs (a) the true excluded and isolated contributions are comparable and (b) ordinary, partial, excluded, and isolated coherence all estimate the true contribution, though excluded and isolated coherence still estimate most accurately (Figure 7D-F and O). In the case of minimally correlated inputs, there was no visually noticeable distinction in coherence vs. frequency plots (Figure 7) between excluded and isolated contribution schemes. Ordinary, partial, isolated, and excluded coherence all estimated the true contribution in this case. This is because cross power becomes negligible in the absence of input correlation, causing all coherence measures to simplify to the special case of ordinary coherence, which accurately represents the true contribution in such a case. If the inputs were perfectly uncorrelated and measurement noise removed, then all coherence measures would be identical. However, given that some

coherence was still present between inputs, excluded coherence still estimated the excluded contribution most accurately and isolated coherence still estimated the isolated contribution most accurately (Figure 8).

5.2 Limitations

The simulations shown here represented only the simple case of a purely linear, 3-input system with uncorrelated noise on the input and output. Not included were the effects of system nonlinearities, feedback, unmeasured correlated inputs, correlated noise, or system size (number of measured inputs), all of which may adversely affect the performance of these novel coherence measures. Future work is needed to investigate how the proposed coherence measures behave under such conditions.

5.3 Conclusions

We have proposed a novel framework of contribution coherence for contribution analysis of multiple-input systems with correlated inputs. This framework stands as a generalization of existing coherence measures. Additionally, we have demonstrated that, when inputs are correlated, excluded and isolated coherence are the only coherence measures that consistently estimate the true contribution of a given input.

5.4 Acknowledgements

This study was funded by NIH grant R15NS087447-02 “Predicting tremor: Developing a validated, subject-specific model of tremor”.

6 References

[1] H. B. Huang, X. R. Huang, M. L. Yang, T. C. Lim, and W. P. Ding, "Identification of vehicle interior noise sources based on wavelet transform and partial coherence analysis," *Mechanical Systems and Signal Processing,* vol. 109, pp. 247-267, 2018/09/01/ 2018, doi: https://doi.org/10.1016/j.ymssp.2018.02.045.

[2] J. Li, M. Ruiz-Sandoval, B. F. Spencer Jr., and A. S. Elnashai, "Parametric time-domain identification of multiple-input systems using decoupled output signals," *Earthquake Engineering & Structural Dynamics,* vol. 43, no. 9, pp. 1307-1324, 2014, doi: https://doi.org/10.1002/eqe.2398.

[3] T. H. Corie and S. K. Charles, "Simulated Tremor Propagation in the Upper Limb: From Muscle Activity to Joint Displacement," (in eng), *J Biomech Eng,* vol. 141, no. 8, pp. 0810011-08100117, Aug 1 2019, doi: 10.1115/1.4043442.

[4] H. Kuang, Y. Qiu, X. Zheng, B. Wan, S. Xiang, and X. Fang, "Identification of steering wheel vibration source of internal combustion forklifts based on wavelet coherence analysis," *Applied Acoustics,* vol. 197, p. 108947, 2022/08/01/ 2022, doi: https://doi.org/10.1016/j.apacoust.2022.108947.

[5] K. Yoo and U.-C. Jeong, "Identification of Automotive Seat Rattle Noise Using an Independent Component Analysis-Based Coherence Analysis Technique," (in English), *Applied Sciences,* Report vol. 10, 2020/10/15/ // 2020. [Online]. Available: https://link.gale.com/apps/doc/A641750987/AONE?u=byuprovo&sid=bookmark-AONE&xid=ad407605.

[6] S.-G. Park, H.-S. Kim, H.-J. Sim, and J.-E. Oh, "Multi-dimensional spectral analysis of the noise contribution from a drum washer with a dehydrating condition," *Journal of Mechanical Science and Technology,* vol. 22, no. 2, pp. 287-292, 2008/02/01 2008, doi: 10.1007/s12206-007-1042-5.

[7] S. Choi, J. U. Jang, S. Lee, and S.-K. Lee, "Identification of the contribution of a coherent vibration source in a gasoline direct-injection engine using a two-input single-output model," *Proceedings of the Institution of Mechanical Engineers, Part D: Journal of Automobile Engineering,* vol. 227, no. 12, pp. 1698-1705, 2013, doi: 10.1177/0954407013501649.

[8] C. Dai, N. L. Suresh, A. K. Suresh, W. Z. Rymer, and X. Hu, "Altered Motor Unit Discharge Coherence in Paretic Muscles of Stroke Survivors," (in eng), *Front Neurol,* vol. 8, p. 202, 2017, doi: 10.3389/fneur.2017.00202.

[9] W. Liu, Y. Zhang, Z.-J. Feng, J.-S. Zhao, and D. Wang, "A study on waviness induced vibration of ball bearings based on signal coherence theory," *Journal of Sound and Vibration,* vol. 333, no. 23, pp. 6107-6120, 2014/11/24/ 2014, doi: https://doi.org/10.1016/j.jsv.2014.06.040.

[10] Y. Qiu and M. J. Griffin, "Transmission of vibration to the backrest of a car seat evaluated with multi-input models," *Journal of Sound and Vibration,* vol. 274, no. 1, pp. 297-321, 2004/07/06/ 2004, doi: https://doi.org/10.1016/j.jsv.2003.05.015.

[11] D. T. Westwick, E. A. Pohlmeyer, S. A. Solla, L. E. Miller, and E. J. Perreault, "Identification of multiple-input systems with highly coupled inputs: application to EMG prediction from multiple intracortical electrodes," (in eng), *Neural Comput,* vol. 18, no. 2, pp. 329-55, Feb 2006, doi: 10.1162/089976606775093855.

[12] J. Bendat and A. Peirsol, *Random Data: Analysis and Measurement Procedures*, 4th ed. (Wiley Series in Probability and Statistics). Hoboken, New Jersey: John Wiley & Sons, Inc., 2010.

[13] J. Cadzow and O. Solomon, "Linear modeling and the coherence function," *IEEE Transactions on Acoustics, Speech, and Signal Processing,* vol. 35, no. 1, pp. 19-28, 1987, doi: 10.1109/TASSP.1987.1165022.

[14] R. Pintelon and J. Schoukens, *System Identification: A Frequency Domain Approach*, 2nd ed. (IEEE Press). Hoboken, New Jersey: John Wiley & Sons, Inc., 2012.

[15] D. B. Free *et al.*, "Hand and distal joint tremor are most coherent with the activity of elbow flexors and wrist extensors in persons with essential tremor," (in eng), *J Appl Physiol (1985),* vol. 136, no. 2, pp. 337-348, Feb 1 2024, doi: 10.1152/japplphysiol.00407.2023.
[16] S. Price and R. Bernhard, "Virtual coherence: A digital signal processing technique for incoherent source identification," in *Proceedings of IMAC*, 1986, vol. 4, pp. 3-6.
[17] R. J. Allemang, R. S. Patwardhan, M. M. Kolluri, and A. W. Phillips, "Frequency response function estimation techniques and the corresponding coherence functions: A review and update," *Mechanical Systems and Signal Processing,* vol. 162, p. 108100, 2022/01/01/ 2022, doi: https://doi.org/10.1016/j.ymssp.2021.108100.
[18] M. A. Guevara and M. Corsi-Cabrera, "EEG coherence or EEG correlation?," *International Journal of Psychophysiology,* vol. 23, no. 3, pp. 145-153, 1996/10/01/ 1996, doi: https://doi.org/10.1016/S0167-8760(96)00038-4.
[19] C. J. Dodds and J. D. Robson, "Partial coherence in multivariate random processes," *Journal of Sound and Vibration,* vol. 42, no. 2, pp. 243-249, 1975/09/22/ 1975, doi: https://doi.org/10.1016/0022-460X(75)90219-9.
[20] B. K. Bae and K. J. Kim, "A Hilbert Transform Approach in Source Identification via Multiple-Input Single-Output Modeling for Correlated Inputs," *Mechanical Systems and Signal Processing,* vol. 12, no. 4, pp. 501-513, 1998/07/01/ 1998, doi: https://doi.org/10.1006/mssp.1997.0152.
[21] D. G. Smallwood, "Using Singular Value Decomposition to Compute the Conditioned Cross-Spectral Density Matrix and Coherence Functions," in *66th Shock & Vibration Symposium*, 1995, vol. 1, p. 109.
[22] S. Malekpour, J. A. Gubner, and W. A. Sethares, "Measures of generalized magnitude-squared coherence: Differences and similarities," *Journal of the Franklin Institute,* vol. 355, no. 5, pp. 2932-2950, 2018/03/01/ 2018, doi: https://doi.org/10.1016/j.jfranklin.2018.01.014.

7 Supplemental Materials

7.1 Cross Component Coherence Bounds

Here we prove the inequality given in Equation (14) in the text:

$$\left|\Gamma_{ijy}(f)+\Gamma_{jiy}(f)\right| \leq \Gamma_{iiy}(f)+\Gamma_{jjy}(f)$$

We begin with the well-known cross-spectrum inequality [12], an application of the Cauchy-Schwarz inequality, which guarantees that the squared magnitude of cross-power spectral density between two signals at a given frequency ($G_{ij}(f)$) is always less than or equal to the product of the power spectral densities of the two signals at that same frequency ($G_{ii}(f)$ and $G_{jj}(f)$):

$$\left|G_{ij}(f)\right|^2 \leq G_{ii}(f)G_{jj}(f) \tag{SM.1}$$

A similar inequality can be constructed using ordinary and cross component coherence, namely, that the squared magnitude of cross component coherence between two input signals at a given frequency ($\Gamma_{ijy}(f)$) is less than or equal to the product of the ordinary component coherence of the two input signals at the same frequency ($\Gamma_{iiy}(f)$ and $\Gamma_{jjy}(f)$):

$$\left|\Gamma_{ijy}(f)\right|^2 \leq \Gamma_{iiy}(f)\Gamma_{jjy}(f) \tag{SM.2}$$

To prove this inequality, we substitute in the definitions for ordinary and cross component coherence (Equation (12)and Equation (13), respectively):

$$\frac{\left|H_{iy}(f)\right|^2\left|H_{jy}(f)\right|^2\left|G_{ij}(f)\right|^2}{G_{yy}(f)^2} \leq \frac{\left|H_{iy}(f)\right|^2 G_{ii}(f)}{G_{yy}(f)}\frac{\left|H_{jy}(f)\right|^2 G_{jj}(f)}{G_{yy}(f)}$$

Dividing out all terms repeated on both sides of the inequality simplifies this inequality down to the simple cross-spectrum inequality (Equation (SM.1)), proving that Equation (SM.2) is always true.

Though true, this inequality is not necessarily useful. For purposes of interpretation, we are interested in the sum of a cross component coherence pair, not the magnitude of a single cross component coherence term. To express this bound in terms of such a pair we first clarify that, as a conjugate pair, the sum of a cross component coherence pair is equal to twice the real part of either term in the pair:

$$\Gamma_{ijy}(f)+\Gamma_{jiy}(f) = 2Re\left(\Gamma_{ijy}(f)\right) = 2Re\left(\Gamma_{jiy}(f)\right) \tag{SM.3}$$

Furthermore, the magnitude of the real part of a complex number is always less than or equal to the magnitude of the complex number. Combining this fact with Equation (SM.3) and twice the square root of (SM.2) gives the result:

$$\left|\Gamma_{ijy}(f)+\Gamma_{jiy}(f)\right| = \left|2Re\left(\Gamma_{ijy}(f)\right)\right| \leq 2\left|\Gamma_{ijy}(f)\right| \leq 2\sqrt{\Gamma_{iiy}(f)\Gamma_{jjy}(f)}$$

Or more simply:

$$\left|\Gamma_{ijy}(f)+\Gamma_{jiy}(f)\right| \leq 2\sqrt{\Gamma_{iiy}(f)\Gamma_{jjy}(f)} \tag{SM.4}$$

Although the inequality is now formulated in terms of an interpretable quantity of interest, the upper bound itself is not easily interpreted. To remedy this, we now prove that twice the square root of a product of two numbers is always less than or equal the sum of the two numbers:

$$2\sqrt{AB} \leq A + B \qquad \text{(SM.5)}$$

Squaring both sides of the inequality gives:

$$4AB \leq A^2 + 2AB + B^2$$

Grouping like terms yields:

$$0 \leq A^2 - 2AB + B^2$$

which simplifies to:

$$0 \leq (A - B)^2$$

which is always true, proving that the inequality in Equation (SM.5) can be applied to Equation (SM.4) to get a final form for a bound on cross component coherence:

$$\left|\Gamma_{ijy}(f) + \Gamma_{jiy}(f)\right| \leq \Gamma_{iiy}(f) + \Gamma_{jjy}(f)$$

This is the bound presented in the text as Equation (14).

7.2 Excluded Coherence Upper Bound - General

Here we prove that excluded coherence ($\Gamma_{-Sy}(f)$) is always less than or equal to multiple coherence ($\gamma^2_{y:x}(f)$):

$$\Gamma_{-Sy}(f) \leq \gamma^2_{y:x}(f) \qquad \text{(SM. 6)}$$

In a q-input system, multiple coherence can be expressed as the sum of all component coherence terms for the full set of inputs:

$$\gamma^2_{y:x}(f) = \sum_{j=1}^{q}\sum_{k=1}^{q} \Gamma_{jky}(f)$$

For a given subset of inputs, X_S, and the compliment to that subset, $X_{S'}$, this summation can be split into two parts: the summation of all components associated with any element of the subset (i.e., j *or* k included in S) plus the summation of all components associated *only* with the complement subset (i.e., j *and* k included in S'):

$$\gamma^2_{y:x}(f) = \sum_{j=1}^{q}\sum_{k=1}^{q} \Gamma_{jky}(f), (j \in S) \vee (k \in S) + \sum_{j=1}^{q}\sum_{k=1}^{q} \Gamma_{jky}(f), (j \in S') \wedge (k \in S')$$

The first summation is equivalent to the excluded coherence for the given subset ($\Gamma_{-Sy}(f)$), and the second summation is equivalent to the isolated coherence for the compliment subset ($\Gamma_{+S'y}(f)$):

$$\gamma^2_{y:x}(f) = \Gamma_{-Sy}(f) + \Gamma_{+S'y}(f) \qquad \text{(SM. 7)}$$

Because isolated coherence is always greater than or equal to zero, excluded coherence must always be less than or equal to multiple coherence.

7.3 Excluded Coherence Upper Bound – Single Input

Here we prove that in the special case where excluded coherence is calculated for a single input ($\Gamma_{-iy}(f)$), its value is further bounded by the ordinary coherence for that input ($\gamma^2_{iy}(f)$):

$$\Gamma_{-iy}(f) \leq \gamma_{iy}^2(f) \tag{SM. 8}$$

This relationship is proved using an alternate formulation of the SI optimum FRF derived using principles of input conditioning presented in [12]. In a system with q inputs, the SI optimum FRF ($L_{iy}(f)$) between an input and the output can be expressed as the sum across all inputs of the product of the SI optimum FRF between the given input and another input ($L_{ij}(f)$, Equation (3)) times the MI optimum FRF between that other input and the output ($H_{jy}(f)$, Equation (5)):

$$L_{iy}(f) = \sum_{j=1}^{q} L_{ij}(f) H_{jy}(f)$$

Substituting this definition into the formula for ordinary coherence (Equation (2)) yields:

$$\gamma_{iy}^2(f) = \frac{\left|\sum_{j=1}^{q} L_{ij}(f) H_{jy}(f)\right|^2 G_{ii}(f)}{G_{yy}(f)}$$

The magnitude-squared term can be decomposed into the product of the summation and its conjugate, which can be expressed as a double summation:

$$\gamma_{iy}^2(f) = \frac{\sum_{j=1}^{q}\sum_{k=1}^{q} H_{jy}^*(f)\, H_{ky}(f)\, L_{ij}^*(f) L_{ik}(f) G_{ii}(f)}{G_{yy}(f)}$$

The original definition of an SI optimum FRF (Equation (3)) can then be substituted in, yielding:

$$\gamma_{iy}^2(f) = \frac{\sum_{j=1}^{q}\sum_{k=1}^{q} H_{jy}^*(f)\, H_{ky}(f)\, \frac{G_{ij}^*(f)}{G_{ii}(f)} \frac{G_{ik}(f)}{G_{ii}(f)}\, G_{ii}(f)}{G_{yy}(f)}$$

which simplifies to:

$$\gamma_{iy}^2(f) = \frac{\sum_{j=1}^{q}\sum_{k=1}^{q} H_{jy}^*(f)\, H_{ky}(f)\, G_{ij}^*(f) G_{ik}(f)}{G_{ii}(f) G_{yy}(f)}$$

Whenever $j = i$ or $k = i$, the given term simplifies to a component coherence term, such that the double summation can be split into two parts:

$$\gamma_{iy}^2(f) = \sum_{j=1}^{q}\sum_{k=1}^{q} \Gamma_{jky}(f)\,, (j = i) \wedge (k = i) \; + \frac{\sum_{j=1}^{q}\sum_{k=1}^{q} H_{jy}^*(f)\, H_{ky}(f)\, G_{ij}^*(f) G_{ik}(f),\;\; (j \neq i) \vee (k \neq i)}{G_{ii}(f) G_{yy}(f)}$$

where terms in the first summation are summed if j *or* k equal i and terms second are summed if j *and* k are not equal to i. The first summation matches the definition for excluded coherence of input i (Equation (15)), and the second can be expressed as the squared magnitude of a single summation:

$$\gamma_{iy}^2(f) = \Gamma_{-iy}(f) + \frac{\left|\sum_{j=1}^{q} H_{jy}(f) G_{ij}(f),\; j \neq i\right|^2}{G_{ii}(f) G_{yy}(f)}$$

Thus, ordinary coherence is equal to excluded coherence for that input, plus some additional term. This additional term is always greater than or equal to zero (because numerator and denominator are non-negative), guaranteeing that excluded coherence for a single input will always be less than or equal to ordinary coherence for that input.