Fundamental Analysis of Dynamic Response in the Presence of Bilinear Stiffness

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ABSTRACT

Many physical structures have joints that have differing stiffnesses depending on the direction of deflection. These joints can be modeled as bilinear, where the stiffnesses is assumed to be linear on each side of a transition point, but significantly greater on one side. This work presents a detailed study of bilinear nonlinearity in single and multiple degree of freedom systems. The logistic function is used to regularize the bilinear function, so that its derivative is always continuous, and the effect of this regularization is studied. In particular, increasing the rate at which the slope transitions is found to cause the nonlinear behavior to activate at lower energies. The systems' properties are then explored, such as its frequency vs amplitude behavior, harmonics, nonlinear normal mode (NNM), and other response characteristics. The results show that bilinearity generally causes the NNM frequency to decrease with increasing amplitude. However, if the point at which the stiffness transitions is moved away from the origin, then hardening, softening and a nearly amplitude-independent frequency are observed. The system tends to exhibit a response that is approximately comprised of two half periods, each of which oscillates at a frequency commensurate with the corresponding stiffnesses for positive and negative displacements. The forced response of the bilinear system can have several higher harmonics that are quite strong. The ratios of the amplitudes of these harmonics to the amplitude of the fundamental frequency increases until

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they saturate. The findings presented can be helpful in identifying bilinear behavior in measurements and predicting failures that linearized models would not exhibit.

Keywords: bilinear, piecewise-linear, nonlinear vibration, stiffness nonlinearity, failure

Introduction

Many structures and mechanisms include components that have differing stiffness depending on the direction of deflection. This can occur because of the geometry of the structure (geometric nonlinearity), structural components contacting other components during motion or opening and closing of cracks or gaps in a component. These characteristics can result from purposeful design, or from a structure being subject to an environment that exceeds its design assumptions, such as a bolted joint opening up as the preload is exceeded. A single degree of freedom system that has a piecewise linear force vs displacement curve with two linear regions can be called a bilinear system. While the behavior is precisely linear in each region, any response that visits both regions will exhibit nonlinearity. As with other forms of nonlinearity, the natural frequency of a bilinear system can change with amplitude, although it is bounded by the frequencies that would be observed if the system were to oscillate entirely in one of the two linear regions. Understanding how bilinearity works is vital to correctly predicting the performance of the many structures that incorporate bilinear elements.

Because linear systems are easier to analyze and design, many designers use linearized models of nonlinear systems to approximate their behavior, often to great success. However, in other cases the linear approximation may be poor or the system may exhibit phenomena that have no linear counterpart. One form of testing that could be affected by this is termed “qualification testing.” This type of testing exposes a part to a vibration environment that, by some metric, is a specified amount higher than the expected level. This testing can be done to ensure that the part survives the expected operational forces, or it can be done to determine the maximum operational forces that the part can experience before failure. In the latter case, this data can help determine if a part that experienced unexpected forcing levels is still safe to use or needs to be replaced.

This paper uses nonlinear modal analysis to seek to understand the phenomena that may be exhibited by a bilinear system. The frequency-amplitude behavior of the system is found to depend on both the bilinear stiffness and where the transition happens relative to the equilibrium position. This understanding is used to explore how the parameters of a single degree-of-freedom bilinear system can be identified from frequency-amplitude measurements. Then, the responses of a variety of bilinear systems are computed in a random vibration environment. The results are used to ascertain when one may achieve acceptable results by treating the system as linear, in one of its two linear limit states, and when this would introduce significant errors. These results should help practicing engineers identify cases when this type of nonlinearity must be accounted for in qualification testing and the associated modeling and data reduction, and when a cheaper, linear model is likely to be sufficient.

Literature review

The definition of bilinearity is not consistent across studies. Some studies define bilinearity in a symmetric manner where the force-displacement curve is linear around the origin and the slope changes at two transition points, one for positive and one for negative displacement, at a specified displacement level [1], [2]. Such a system could be termed trilinear, as there are actually three linear regions, and the response tends to resemble that of a cubically nonlinear system; asymmetry is typically not present when using that model. Other studies, including this one, define bilinearity in an asymmetric manner where two linear regions with different slopes meet at a single transition point [3]–[12]. Among these asymmetric bilinearity studies, some limit their focus to models where the transition occurs at zero displacement [8], [10], [11], while others allow for the transition point to occur at non-zero displacement [3], [5], [7], [9], [12]. Throughout this study, transitions that occur at zero displacement are referred to as centered transitions, while transitions that occur at non-zero displacement are referred to as uncentered transitions.

Bilinear stiffness in a system causes several nonlinear behaviors. Wong, et al. [12] used the incremental harmonic balance method to determine all possible superharmonics and subharmonics of a bilinear system. Multiple solutions and chaos are also observed in these systems, as can be seen in [13].

Several different approaches have been developed to model bilinearity. In many studies the system is piecewise linear and the system experiences an instantaneous change in stiffness. Shaw and Holmes in [4] showed that the time response of an uncentered bilinear system can be found analytically by finding the roots of transcendental equations. To avoid this complication, many studies instead focus on approximation methods. Shaw and Holmes used Poincaré maps, bifurcation analysis, and orbit plots to analyze the system [4]. Several studies use numerical time integration of the equations of motion, if not as a primary analysis tool, as truth data that is compared against (e.g. [11], [13], [14]). This can be very time consuming because of the instantaneous transition. When the response crosses the transition point, adaptive algorithms must iterate many times to capture the transition sufficiently. If too large a timestep is used across the transition, the system can penetrate farther into the stiff region than it should, causing integration errors to accrue. Saunders, et al. analyzed the effects of accurately capturing the transition timing in [14]. In other papers, various methods are used to approximate the response, such as the harmonic balance method [11] and the incremental harmonic balance method [12].

Some methods of analysis require the transition in a bilinear system to be approximated or regularized with a differentiable function. For example, the method proposed by Kerschen and Peeters to calculate the Nonlinear Normal Modes (NNMs) of a system requires the derivative of the force vs displacement curve at all possible displacements [15]. Several functions have been used to make this regularization. Saunders, et al. [13] compared five different methods of regularizing the transitions for a symmetric bilinear system. Some of these models include polynomials in a power series representation, rational polynomial functions, and hyperbolic tangent functions or arctangent functions. For the power series to converge to the piecewise linear model, it requires a large number of terms in the summation. The hyperbolic tangent and arctangent functions have a single parameter that controls how converged the approximation is to the piecewise linear model. Christopher, et al. used what they called a “sigmoid function” to model asymmetric bilinear behavior [16]. This function could also be called a “logistic function” as in [17]. In general, logistic functions are a particular type of sigmoid function, so these terms are sometimes used interchangeably. This function uses an exponential function in the denominator of a rational function. Similar to the tangent type functions, this function has a single convergence parameter controlling the model accuracy relative to the piecewise linear model. Though not explored in this work, the type of regularization used can affect the nonlinear behavior that is observed [13]. This work uses a logistic function to regularize the bilinear stiffness. Having a single parameter controlling how closely the logistic function approximation matches the bilinear model is much simpler than having a polynomial with several parameters that must be carefully selected to give the best possible representation. As the convergence parameter of the logistic function becomes large, accounting for numerical overflow might become necessary. In this work it is assumed that the logistic function regularization will have similar advantages and disadvantages as the hyperbolic tangent and inverse tangent functions explored in [13].

The frequency at which a centered bilinear stiffness system oscillates can be approximated using the bilinear frequency approximation (BFA) [4], [6]. This approximation assumes that a full cycle consists of two half cycles, each at the linear frequency of each of the two stiffnesses. The amplitude of oscillation can be approximated using the bilinear amplitude approximation (BAA) in [9]. This approximation calculates the vibration modes at both stiffness values, makes a matrix with the modes of both cases included as columns, and calculates the singular value decomposition of this matrix. One or two left singular vectors associated with the largest singular values are used as the modes of the bilinear system. This method was extended to systems with uncentered transitions in [3]. These methods only apply to SDOF systems, and they focus primarily on time domain responses. Because of these limitations, these methods do not give a general picture of the global behavior of a bilinear system. Thus, additional methods of analysis that capture the more general aspects of the system response would be beneficial.

One method engineers use to analyze nonlinear systems is nonlinear normal modes. The nonlinear normal modes of bilinear systems with an infinitely sharp transition were analyzed in [7] using an invariant manifold approach. These NNMs started at the linear frequency corresponding to whichever side of the transition the origin was located. The frequency then suddenly started changing once the amplitude reached the transition point. This sharp point in the NNM is a result of the uncentered transition point that is also infinitely sharp. If the transition point on the force vs displacement curve was regularized instead of infinitely sharp, then the change in resonant frequency on the NNM would be gradual instead of sharp. While this smoothing of the transition is an approximation of the piecewise-linear model, this smoothing may better reflect the stiffness that actually occurs in many physical bilinear systems. Thus, regularized models may provide computational advantages while also approximating physical structures more accurately.

In qualification testing, the amplitude of vibration is increased to a certain level higher than the expected use amplitudes. In [13], [14], [16], piecewise linear systems are simulated with increasing amplitudes of vibration. They show nonlinear behavior such as subharmonics, superharmonics, and chaotic behavior occur. Using a linear model during similar qualification tests could result in inaccurate test results or unexplained behaviors. These discrepancies could result in overconservative or underconservative results, either of which could be beneficial or detrimental depending on the testing case.

This work further studies bilinear stiffness models, including both centered and uncentered cases. A regularized force vs displacement function is used to model the bilinear springs. Various nonlinear behaviors are observed using nonlinear normal modes (NNMs) and power spectral density (PSD) plots. The NNMs will have some differences compared to those observed in [7] due to the regularization of the bilinear model. A novel focus of this work is how stiffness bilinearity affects failures that occur in parts in a random vibration environment compared to those predicted by linear models.

Methods

The simplest bilinear stiffness could be modeled with a step function, which when integrated would produce force displacement behavior with two slopes. Alternatively, to ensure a smooth transition between the two stiffnesses, the stiffness can instead be modeled using a logistic function,

$$y(x) = \frac{1}{1 + e^{-x}} \tag{1}$$

where $y(x)$ is the stiffness as a function of the displacement x.

This type of function was used by Christopher, et al. in [16] to model intermittent contact with a stop/structure. It is often used in machine learning to fit models that separate data into binary categories [17]. This function allows for a smooth bilinear stiffness function, and it can be modified such that it has a transition that is as sharp as needed to approximate the system of interest.

To allow for modeling a general bilinear spring, several parameters are introduced to equation (1), including the two linear stiffnesses k_n and k_p, a parameter r for how quickly the transition occurs, and an offset parameter c to account for an uncentered transition. The subscripts p and n on the stiffnesses refer to the value associated with positive and negative displacements, respectively. The bilinear stiffness equation is thus

$$k_{bilin}(x) = k_n + \frac{k_p - k_n}{1 + e^{-rx+c}}. \tag{2}$$

If the limit is taken as $x \to \infty$, then $k_{bilin} \to k_p$. If $x \to -\infty$, then $k_{bilin} \to k_n$. The stiffness function is integrated to obtain the bilinear force vs displacement curve

$$f_{nl}(x) = k_p x + \frac{k_p - k_n}{r} \ln\left(\frac{1 + e^{-rx+c}}{1 + e^{c}}\right), \tag{3}$$

where the constant of integration was chosen such that the curve passes through the origin. Incidentally, this process could also be followed to obtain a multilinear function such as the symmetric bilinearity in [13], [14]. Equation (3) regularizes the transition between the two linear stiffnesses. This regularization results in characteristics that are not seen in an unregularized bilinear model, but the unregularized model can be approached as the rate of transition approaches infinity. In real world cases, where the transition is not instantaneous, this regularization allows for more accurate representation of the transition region. In cases where instantaneous transitions occur, a large enough transition rate could be selected to obtain the desired level of accuracy. Some methods of regularization can result in a transition that is not monotonically changing. The model used in this study does monotonically change stiffness values, but the transition in stiffness values occurs exponentially instead of instantaneously. An example is shown in Figure 1 where the stiffness is shown transitioning from 0.5 to 2 at $x = 0$. The stiffness is shown in the top plot and the force vs displacement curve is shown in the bottom plot. The red curve has a transition rate of $r = 1$, which means the transition is relatively slow or over a larger range of displacement. The blue curve with a transition rate of $r = 10$ has a much smaller range of displacement where the transition seems to occur. The corresponding curve on the force vs displacement plot has a much sharper transition and the slopes appear linear at much smaller displacements.

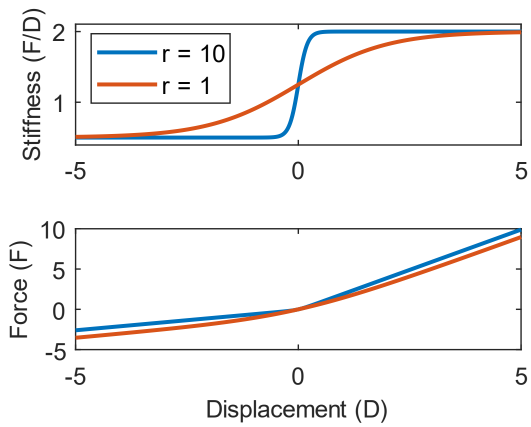

Figure 1. Stiffness and force vs displacement for a bilinear spring using two different transition rates.

In the equation (3), the parameter c is nondimensionalized with respect to r in the sense that a particular value of c always represents a particular percentage of slope transition occurring by the zero-displacement point. To shift the center of the transition point to a specific displacement, then the substitution $c = rx_i$ can be made. Here, x_i is the displacement at which the center (50%) of the transition occurs. If desired, an additional c could be added to the shifted transition point resulting in an exponential term such as $e^{-r(x-x_i)+c}$. Though not used in this study, this formulation can account for a nominal transition displacement given by x_i and an error or uncenteredness value c.

Because this function is smooth, it can be implemented in nonlinear normal mode (NNM) continuation software that requires knowing the derivative of the force vs displacement relationship at all points [15]. It is challenging to integrate the force displacement relationship algebraically, so the potential energy stored in the bilinear springs is calculated using numerical integration.

With the formulation of a bilinear spring established, the systems to be studied are now discussed. This work studies an SDOF system. The relative simplicity of this systems makes it easier to analytically determine their behavior as the parameters are changed and to tune their natural frequencies to explore a range of phenomena. With a larger number of degrees of freedom this would probably need to be done entirely with numerical methods. This paper focuses on the SDOF system, to seek to thoroughly cover all the phenomena that can be observed in a bilinear SDOF system, or in a single uncoupled mode of a nonlinear MDOF system.

The single degree of freedom model that is studied is diagrammed in Figure 2. It consists of a single mass connected to ground by a bilinear spring. According to the diagram, we assume spring extension is the stiffer side of the spring. The direction of the bilinear spring is arbitrary in this case because changing it will only mirror the results. In cases where damping is used in calculations, a linear damper is included with a damping ratio of 0.00088, which was chosen to mimic a particular system of interest.

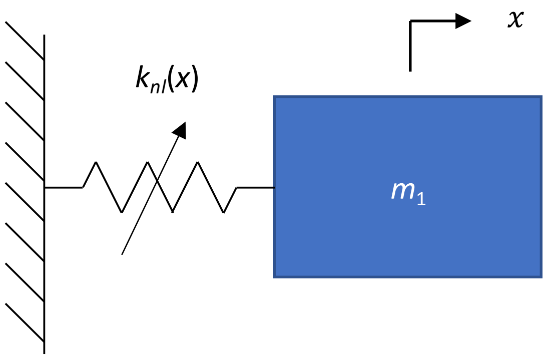

Figure 2. Diagram of SDOF system used in this study. It consists of a single mass connected to ground with a bilinear spring.

The frequency of oscillation of the SDOF bilinear system at low amplitudes can be modeled as

$$\omega_{lin} = \sqrt{\omega_n^2\left(\frac{e^c}{1+e^c}\right) + \omega_p^2\left(\frac{1}{1+e^c}\right)}, \tag{4}$$

where $\omega_p = \sqrt{\frac{k_p}{m}}$ and $\omega_n = \sqrt{\frac{k_n}{m}}$. This equation is found by setting $x = 0$ in equation (2) and dividing by the mass to find the frequency.

The frequency of oscillation of the SDOF bilinear system at high amplitudes can be modeled as

$$\omega_{bilin} = \frac{2\omega_p\omega_n}{\omega_p + \omega_n}. \tag{5}$$

This equation is derived by assuming a full period of oscillation is composed of two half periods at each of the two linear frequencies. This was first presented in [4] and is known by some as the “bilinear frequency” (see section 5 in [6]) as well as the “bilinear frequency approximation” (see [3]). Incidentally, equation (5) is the harmonic mean of the two linear frequencies ω_p and ω_n. This equation applies to cases with uncentered transitions as well because, as the amplitude of the oscillation increases, the distance from the origin to the transition becomes small in comparison to the total amplitude.

Results

The SDOF system will be analyzed using NNMs and then PSDs of various types of random forcing. Possible failure modes of the SDOF system will be discussed. The stiffnesses for the positive and negative displacements are assumed to be significantly different here to highlight the observed phenomena. It is assumed that the positive displacement stiffness is 10 times the negative displacement stiffness. While many systems may have less dramatic differences in positive and negative stiffness, this would simply result in the observed behavior being less dramatic than will be seen here.

The general shape of the SDOF NNM is shown in Figure 3.

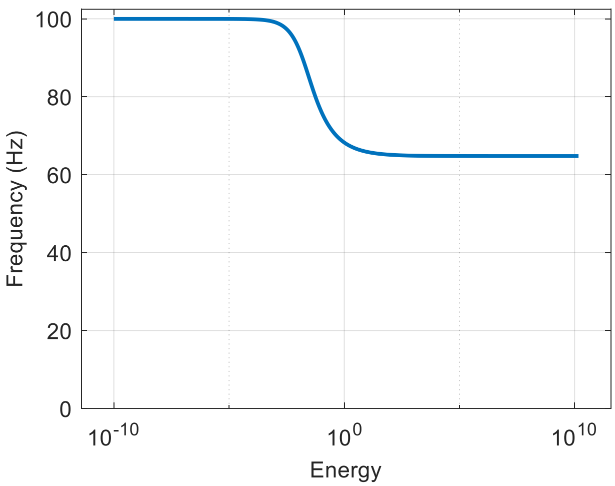

Figure 3. Nonlinear normal mode (NNM) branch of the SDOF system being studied.

The NNM starts at the linear frequency at low energy levels. As the energy level increases, the frequency starts to shift and asymptotically approaches the bilinear frequency. Many types of nonlinearity result in NNMs whose frequency continues to increase or decrease indefinitely after the nonlinearity is activated. The bilinear system is different in that its frequency has a limit or asymptote at high energy. The low energy frequency is determined by c, ω_n, and ω_p. The equation for this frequency is equation (4). The high energy frequency, or bilinear frequency, is determined by ω_n and ω_p using equation (5).

The NNM transitions from the low energy frequency to the high energy frequency at an energy level determined by the transition rate r and the offset value c. If r is increased (the transition made sharper) then the frequency shift occurs at lower energy levels. This occurs because the regularization creates an amplitude range where the low amplitude oscillations are essentially linear, and larger r values result in a smaller linear range. For bilinear behavior to be observed, the amplitude must be large enough to escape this range. If $c = 0$, then in the limit as r approaches infinity the transition to bilinear behavior no longer exists, and the NNM is always the bilinear frequency. This constant frequency behavior is observed in [4]. For all values of r, the value of c can delay the frequency transition, where a larger absolute value of c results in the transition occurring at a higher energy level. This type of frequency shift delay still occurs when the transition is infinitely sharp. In this case, the linear frequency is the frequency associated with whichever stiffness occurs at the origin, and once the amplitude is large enough for the transition to occur, the frequency starts shifting toward the bilinear frequency. This is seen in [4], [7]. Having c as a parameter allows for the linear frequency to be virtually anywhere between the frequencies associated with the stiff side stiffness and the soft side stiffness. This means that the natural frequency of the system could start below the bilinear frequency and increase up to it. One interesting case is when c is chosen such that the low energy frequency is the same as the high energy frequency. When this is occurs, the frequency has a small drop when the transition is reached, but the frequency then rises back to the desired frequency as seen in Figure 4.

In experiments [18] bilinear systems are often observed to exhibit a negligible change in natural frequency as amplitude increases. The survey above reveals that this can happen for one of two reasons. The value of c may have been chosen as shown in Figure 4, or the system may transition instantaneously to the saturated region, governed by equation (5). In the latter case, the system's time response will exhibit nonlinearity, as detailed later.

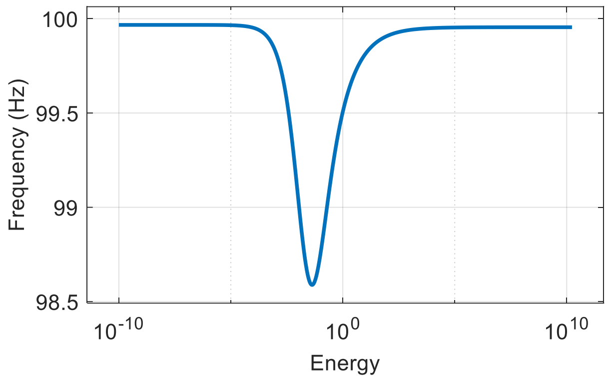

Figure 4. SDOF NNM branch when c is chosen such that the low energy frequency and the high energy frequency are equal.

With the parameters c, ω_n, and ω_p, the NNM branch can be shaped to get a wide range of frequency vs energy level behavior. Instead, the low energy frequency ω_{lin} and the high energy frequency ω_{bilin} could be used as design parameters. Introducing an additional design parameter for the stiffness proportionality constant $\alpha = \frac{\omega_p^2}{\omega_n^2}$, we can solve for ω_n and ω_p using equation (4). This gives

$$\omega_n^2 = \frac{\omega_{lin}^2}{1 + \frac{\alpha - 1}{1 + e^c}} \tag{6}$$

$$\omega_p^2 = \alpha \omega_n^2 \tag{7}$$

The values for ω_n and ω_p can be also determined using ω_{bilin} instead of α by solving equations (4) and (5), but the equations are more complicated. Numerical solutions of these equations could also be used in this case. Using these equations, an NNM branch for an SDOF bilinear system could be tailored to meet design needs (within the design space set by the equations).

The value for r can be chosen based on the displacement interval in which the stiffness transition needs to occur. If the transition interval size and the percentage of transition in that interval, or d and p respectively, are known, then the required r value is given by

$$r = \frac{2 \ln\left(\frac{1 + p}{1 - p}\right)}{d} \tag{8}$$

where $0 \leq p < 1$. Generally, increasing p or decreasing d results in the natural frequency shift occurring at lower energy levels.

The change in frequency in the NNM branch is accompanied by a change in the shape of the oscillation. At low energy, the oscillation is sinusoidal. At high energy, the displacement in the stiff direction is smaller than in the soft direction. The phase plot of the NNM at high energy looks like an egg shape instead of an ellipse. One period of the NNM at a high energy level and its phase plot can be seen in Figure 5.

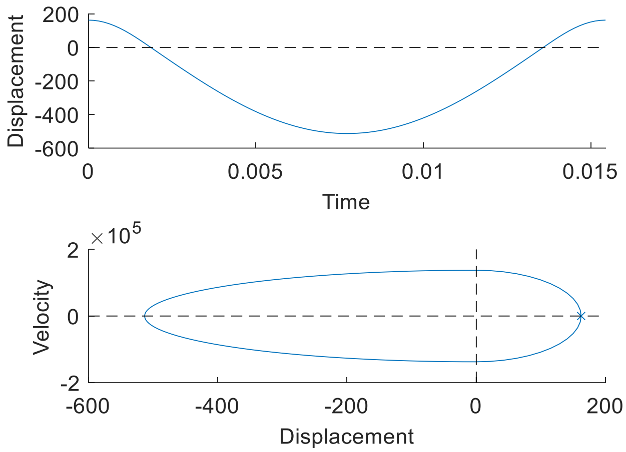

Figure 5. NNM at an energy level of approximately 10^{10}. The top plot shows the NNM in terms of displacement vs time, and the lower plot shows the NNM in terms of the phase plot.

This type of reponse occurs regardless of the values for r and c as long as the amplitudes are sufficiently large. The Fourier transform of a response such as that shown above would contain many harmonics of the fundamental frequency, both even and odd.

Next, the frequency response of the SDOF system is analyzed by simulating the system's response to broadband random forcing. Each time a random forcing time history was generated, the random number generator was reset to the default starting state so that each case would be more directly comparable. The power spectral density (PSD) plot of the system's response is shown in Figure 6.

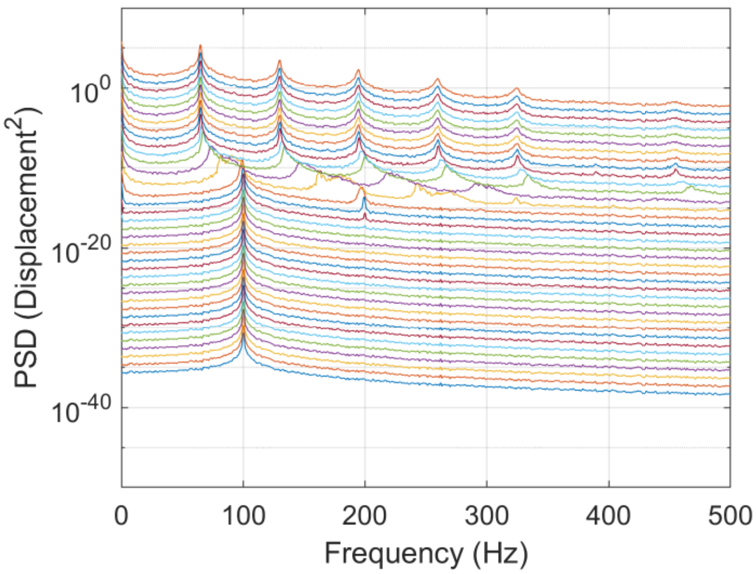

Figure 6. Power spectral density plot of the SDOF system subject to broadband random forcing. The lowest curve corresponds to the lowest forcing amplitude, and each higher curve had a higher forcing amplitude than the last.

At low amplitudes, the response has a single peak where the linear natural frequency occurs. As the amplitude increases, two things are observed: first, the frequency at which the fundamental peak occurs shifts in accordance with the NNM; second, several harmonics appear, in this case indicating that the response is becoming asymmetrical due to the bilinear stiffness. The harmonics also shift proportionally to the fundamental frequency shift.

Failure of a system is typically governed by the root-mean-square (RMS) displacement or stress, which is the area enclosed by the PSD curve. It is informative to compare this to that of the low-level linear system, to see whether the bilinear system is more or less likely to experience failure. This is shown in Figure 7. Surprisingly, the results show that the RMS response increases nearly linearly in proportion to the forcing amplitude, exactly as happens for a linear system (solid line in the figure). This reveals that, when performing failure analysis for a bilinear system, one could approximate the system as linear with very little loss in accuracy. In the context of qualification testing, one can use linear techniques to determine the margin to failure. However, it should be noted that this analysis does not take into account any difference in strength between the positive and negative stiffness regions; if those regions have significantly different strengths then one would need to decompose the RMS displacement into positive and negative components (i.e. see Figure 5) and apply the strength criteria to each.

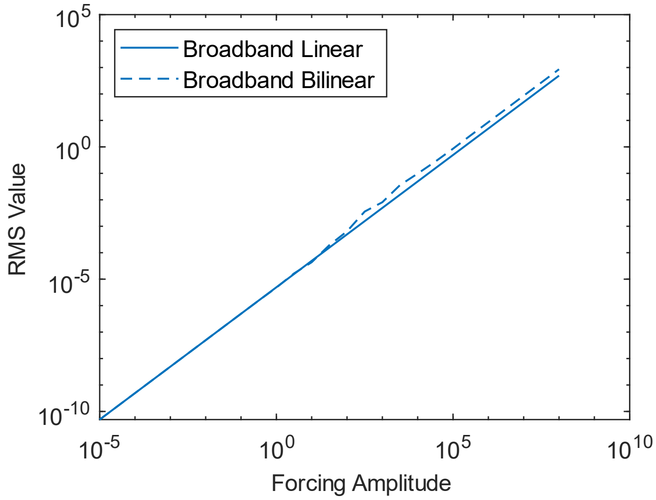

Figure 7. RMS of the displacement of the linear and bilinear SDOF systems subject to broadband random forcing.

This finding begs the question, are there cases in which a bilinear system may exhibit failures at force levels that are significantly different from those exhibited by a linear system? It was discovered that this can occur when the SDOF system is excited by a bandpass filtered random forcing. For example, first consider a forcing that is centered below the first resonance. In this case the forcing is confined to the band of 20-30 Hz, for the system whose NNM is shown in Figure 3 to transition from 100 Hz at low amplitude to 60 Hz at high amplitude. The resulting PSD plot is shown in Figure 8. The low energy plots are affected as expected: the general shape of the linear SDOF mode is present, and the portion inside the forcing frequency band has much higher response amplitudes.

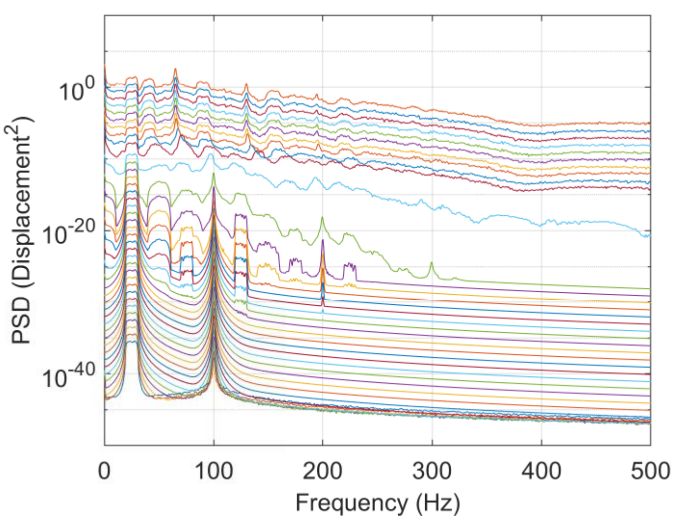

Figure 8. PSD plot of SDOF system when random forcing is bandpass filtered to 20-30 Hz.

However, as the forcing amplitude increases, additional peaks appear. A band of response also appears at twice the frequency of the forcing, from 40-60 Hz. The mode of the system is also excited in all cases, even though the response is focused away from resonance. It is also interesting to note that side-lobes of response occur on both sides of the natural frequency. These bands are 70-80 Hz and 120-130 Hz and correspond to the linear natural frequency plus or minus the forcing frequencies. As the forcing amplitude continues to increase, higher harmonics appear, each with their own bands around them. Eventually the results can get somewhat complicated and messy. The peaks that are generated at high energy levels seem to be explained using a convolution model. This model would be used by taking the linear SDOF PSD response, convolving it with itself, multiplying the result by some decay factor, and adding it to the original PSD. The convolution model explains the natural frequency harmonics, the 0 Hz peak, the forcing band harmonics, and the bands around each natural frequency that look like an amplitude modulation signal.

If the forcing bands coincide with any of the harmonics, the other harmonics seem to be excited as well. This can result in energy being distributed to harmonics that are both higher and lower than the excited harmonic. An example of this is shown in Figure 9 where the forcing frequencies are limited to a 10 Hz window centered on the fully shifted second harmonic. Alternatively, if harmonics remain far enough outside of the forcing band, then the peaks are much smaller and the PSD plot becomes somewhat smoother as seen in Figure 8.

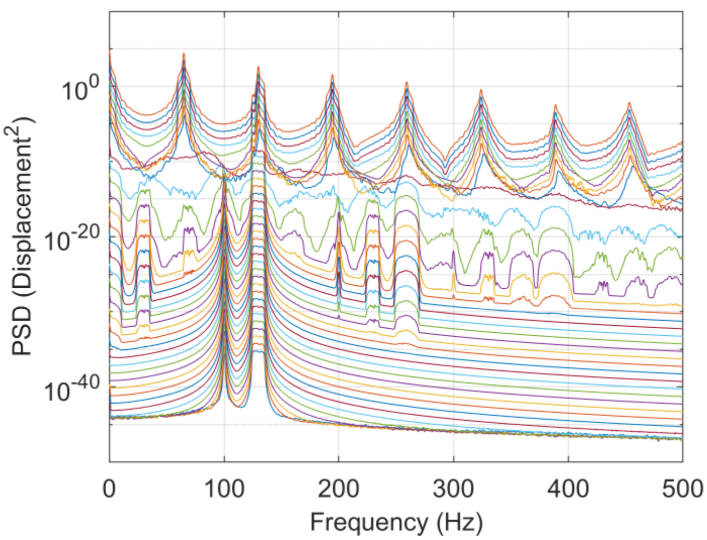

Figure 9. PSD plot of SDOF system with random forcing bandpass filtered to 124.6-134.6 Hz. This frequency band corresponds to a 10 Hz window centered on the fully shifted second harmonic. Several of the high amplitude peaks shown are aliased from higher frequencies.

The nature of the forcing also has a strong influence on the RMS response, and hence on failure. Figure 10 and Figure 11 show the RMS response of the system for two cases, one in which the random forcing is band limited to 125-135 Hz (around the second harmonic of the high energy NNM frequency) and one that is limited to 95-105 Hz (centered on the first linear resonance). In both cases, the RMS displacements increase linearly at low amplitudes; a linear model would capture this behavior well. At high enough forcing amplitude the RMS values can increase above or decrease beneath the linear case. If the harmonics end up within or close enough to the forcing band, then the RMS amplitude increases above the linear case (see Figure 10). If none of the harmonics end up within the forcing band, then the RMS amplitude will drop below the linear case (see Figure 11).

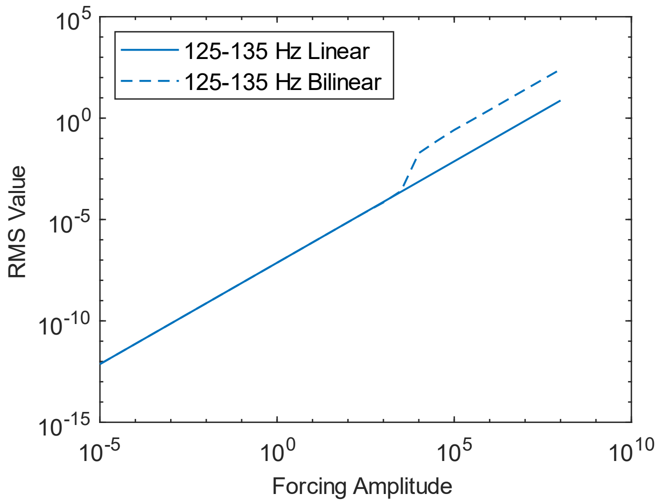

Figure 10. RMS of the displacement of the linear and bilinear SDOF systems subject to bandpass filtered random forcing at 125-135 Hz.

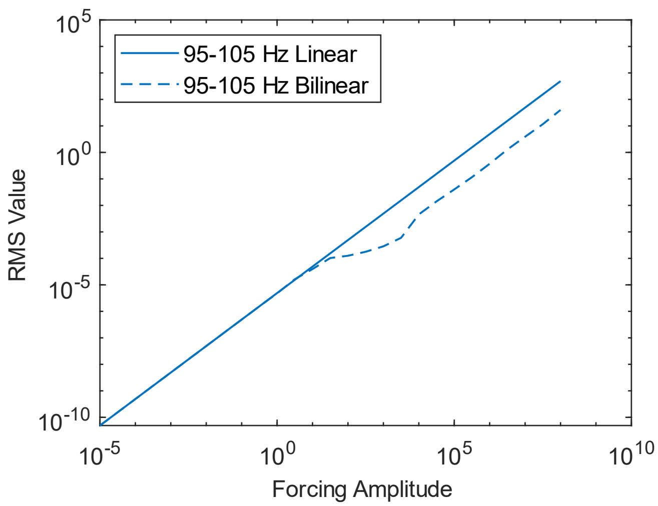

Figure 11. RMS of the displacement of the linear and bilinear SDOF systems subject to band pass filtered random forcing at 95-105 Hz.

Discussion of Failure Modes

When the SDOF system studied here is excited with broadband random forcing, Figure 7 shows that the bilinear system tends to have roughly the same RMS value as the linear system. At high enough amplitudes where the bilinear effects become important, some deviations exist, but they are relatively small and generally increase in the same manner as the linear system. Thus, when a system is forced with broadband forcing, the stiffness bilinearity has little effect on the RMS amplitude. Alternatively, when a system's forcing is random and bandpass filtered, the RMS values can change noticeably once the bilinear effects become significant. In Figure 11 and Figure 10, the RMS value drops significantly and rises significantly, respectively. In the drop occurs because the as the peaks shift to lower frequencies, the peaks leave the forcing band and remain relatively far from it. The rise occurs because of a harmonic shifting into the forcing band, as seen in Figure 9. Thus, once the bilinearity effects become significant, a bilinear system could result in significantly larger or smaller response than that predicted by a linear analysis. Therefore, bilinearity is important to account for when the system of interest is forced by a window of frequencies.

The case represented in both Figure 9 and Figure 10 also highlights an interesting result for the frequency content of the system. In this case, the second harmonic of the fundamental frequency shifted into the forcing band. The RMS plot shows that forcing around the harmonic can lead to significant response levels as compared to the linear case. This becomes more interesting when the PSD plot is examined. As can be seen in Figure 9, forcing the system around the second harmonic resulted in system response around the fundamental frequency as well as higher harmonics. This means that avoiding a response at the fundamental frequency will require avoiding forcing at the higher harmonics as well, not just the fundamental as would be the case for a linear system. Even when the harmonics are avoided as much as possible, Figure 8 shows that a lot of the energy can still be spread to other frequencies than that of the forcing band. Therefore, bilinearity is important to account for when one is seeking to design a system to avoid exciting specific modes or frequencies. In contrast, if the system is subjected to broadband random forcing, the bilinear system has about the same propensity for failure as a linear system.

Another way that the SDOF system could fail arises from the asymmetry of the system. Because one side has a softer stiffness and the other has a stiffer stiffness, the displacements will be larger or smaller than predicted by a corresponding linear system. Larger displacements could result in strain values being larger than allowable. Errors in max displacement predictions could also cause failures when displacements must be limited to or excited above a specific level. If certain clearances in machines are exceeded because of one side being softer than expected, the machines' parts could be damaged and operation could cease. Alternatively, if the deflections had to be larger than a certain limit, the stiff side deflections might be too small resulting in the machine not accomplishing its purpose. Thus, anytime the maximum displacements have lower or upper bounds, accounting for stiffness bilinearity will be important.

Another way the asymmetry of the oscillation could cause failure is by affecting the fatigue life of the system. The mass of the system will spend more time on the soft side of the oscillation than the stiff side. This results in the effective center of oscillation being moved toward the soft side. This results in a time-averaged mean stress, and if it is a tensile mean stress then the fatigue life of the system could be reduced. Thus, stiffness bilinearity should be accounted for when fatigue life of the bilinear element is important.

System Identification

The parameters necessary to define the SDOF bilinear system include 1) the linear/low-amplitude frequency, 2) the ratio of the stiffnesses on each side of the transition, 3) the offset parameter describing how the linear frequency is biased toward one or the other linear frequencies, and 4) the transition rate that describes how much displacement is required to reach a certain amount of stiffness transition. Measuring these parameters on a physical system would allow for this model to be fit to the physical system to determine if bilinear stiffness is the type of nonlinearity occurring and to predict the response.

To measure the linear stiffness, the methods of a linear system can be used provided the amplitudes are sufficiently small so that the nonlinear effects are minimized. The ratio of the two stiffnesses can be measured if the amplitudes are sufficiently large so that the nonlinearity is fully exited. If the nonlinearity can be fully exited, the offset bias can be measured using a plot of the stiff frequency deviation to the soft frequency deviation plotted against either the soft or stiff frequencies. The transition rate can be determined by examining at what energy the frequency changes in the NNM, though this is also affected by the offset bias. Alternatively, the frequency deviation plots can be matched to determine the transition rate. Both methods require the energy of the model and the associated structures energy to be calculated and matched. This could be challenging because of needing to determine equivalent masses and equivalent velocities of the structure.

The stiffness ratio, offset bias, and transition rate could be measured statically. The force vs displacement curve of a structure could be found using static force testing. Once the two linear stiffnesses are approximated, the ratio between them can be calculated. The 50% transition displacement point can be found using this data as well, which can be converted to an offset bias value. The transition can also be fit by ensuring the sharpness of the model transition matches that observed in static testing.

One may also be able to identify the ratio of the positive and negative stiffnesses from the strengths of the harmonics in response to a broadband input. For example, using the response in Figure 6, the magnitude of each peak in the PSD was found and normalized to the magnitude of the fundamental. Figure 12 shows the normalized magnitudes as the forcing amplitude increases. A saturation effect is seen where each of the normalized peaks grows and approaches an asymptote as the forcing amplitude is increased. These high energy limits of the harmonics are affected by the value of α. Larger values of alpha result in larger normalized magnitudes. Harmonics such as these are typically readily visible in the response of a bilinear system. Hence, one could presumably use these ratios to determine the value of α for a system.

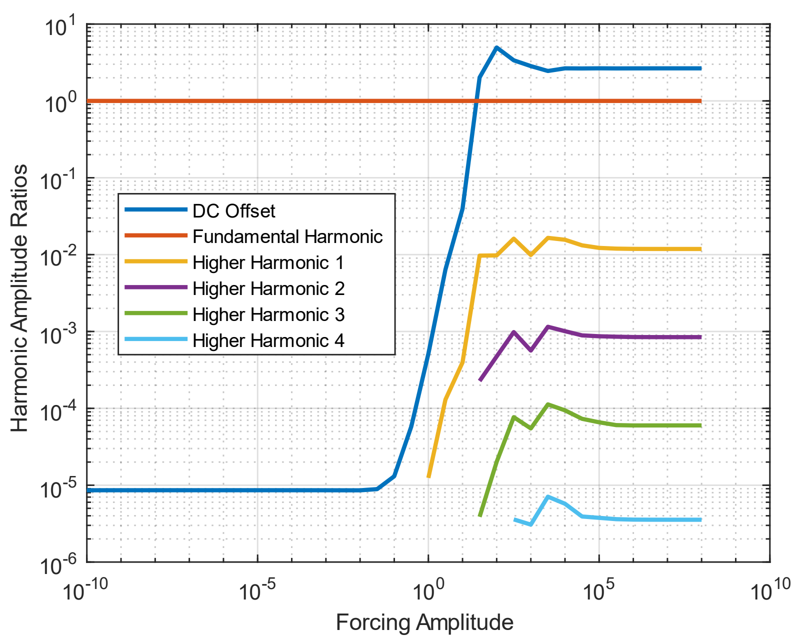

Figure 12. Plot of the magnitudes of each harmonic relative to the fundamental. The DC offset line refers to the PSD value at 0 Hz. Each peak stabilizes at a particular value relative to the magnitude at the fundamental frequency. Higher harmonics have less data at low forcing amplitudes because the peaks are not distinguishable from the noise.

Conclusion

In conclusion, the response of an SDOF system with stiffness bilinearity has been examined in terms of the nonlinear normal mode and the response to random forcing. The NNM has been characterized such that the NNM can be designed to meet the needs of the system designer, within the limits of the equations given. In the bilinear case, the NNM will start at a low amplitude natural frequency and then transition to a different natural frequency at high enough amplitudes. The frequency content of the system response includes higher harmonics that are not seen in the linear case. Once the response amplitude is large enough, the RMS value for the bilinear system can be larger or smaller than the corresponding linear system with the same natural frequency. These results can result in failures due to changing natural frequency, undesirable frequency content, larger or smaller RMS response, larger or smaller asymmetric displacements, and shorter fatigue life. The practicing engineer should compare their system against the failure cases described above to determine if their system model must account for the behaviors produced by bilinearity or if a linear model is likely sufficient for their modeling needs.

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