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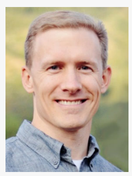
*Submission number: #12381

A Gaussian Process Regression Reduced Order Model of Geometrically Nonlinear Structures



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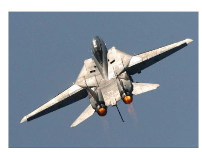
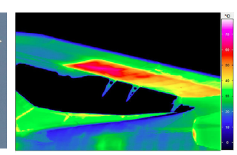
1

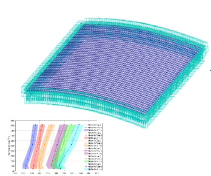
Digital Twin of Geometrically Nonlinear Structures

- Thin structures of high-speed vehicles exhibit highly nonlinear dynamics when subjected to severe aerodynamic or aerothermal stress
- The FE methods have been well-established to simulate the nonlinear dynamics
  - Time history responses
  - Frequency response functions (FRFs)
  - Power spectral densities (PSDs)
  - NNMs
- However, the computational overhead for integrating the nonlinear response becomes prohibitive as FE models (FEMs) become large
- Brings a need of a **Reduced Order Model**

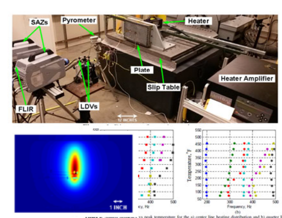




Advanced vehicle under extreme aerothermal stress



FEM/FEA[1]



Experiment[2]

[1] K. Park, and M. S. Allen., Proceedings of IMAC 39th, 83-93. 2021
[2] D. A. Ehrhardt, et al., MAC 36th, 2018

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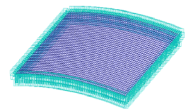
2

Reduced Order Modeling of Geometrically Nonlinear Structures

- Reduce Order Model (ROM) is an alternative of FEM to dramatically reduce the computational expense for computing nonlinear responses of GNL structures
- Significantly reduces the degrees of freedom (DOF) by projecting the full order equations of motion onto a smaller modal subspace
- Many studies have revealed that ROMs can accurately simulate the dynamic responses
  - Enforced Displacement (ED)[3]
  - Implicit condensation and Expansion (ICE)[4]

- **FEM**: $\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{f}_{nl}(\mathbf{x}) = \mathbf{f}(t)$

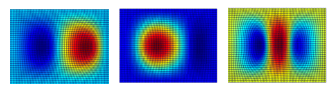


ICE

Galerkin projection using static load-displacement solutions

- **ROM**: $\ddot{q}_r + 2\zeta\omega_n\dot{q}_r + \omega_r^2 q_r + \theta_r(q_1, q_2, \dots, q_n) = \boldsymbol{\phi}_r^T f(t)$

$$\theta_r = \sum_{i=1}^{m}\sum_{j=i}^{m} A_r(i,j)\, q_i q_j + \sum_{i=1}^{m}\sum_{j=i}^{m}\sum_{k=j}^{m} B_r(i,j,k) q_i q_j q_k$$

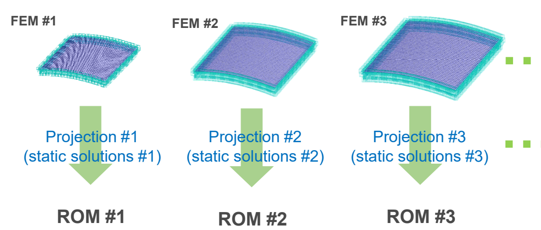


Modal subspace

[3] McEwan, M. I., Wright, J. R., Cooper, J. E., and Leung, A. Y. T., Journal of Sound and Vibration, 2001
[4] Hollkamp, J. J., and Gordon, R. W., Journal of Sound and Vibration, 2008

Issues on Reduced Order Modeling of GNL structures

- A ROM only represents a single FEM configuration
  - ROM does not account for the variation of FEM parameters
  - As FEM parameter changes, the ROM must be re-computed through a Galerkin projection using a new set of static load-displacement data
  - ROM-based dynamic simulations become tedious



$$\ddot{q}_r + 2\zeta\omega_n\dot{q}_r + \omega_r^2 q_r + \theta_r(q_1, q_2, \dots, q_n) = \boldsymbol{\phi}_r^T f(t)$$

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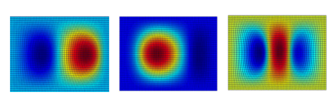


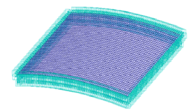
Modal subspace

Issues on Reduced Order Modeling of GNL structures (cont'd)

- ROM is composed of redundant nonlinear coefficients
  - The number of nonlinear coefficients increases cubically as the number of mode increases in ROM
    - e.g.) 2DOF ROM: 14 coefficients
    - e.g.) 3DOF ROM: 48 coefficients

      …
    - e.g.) 7DOF ROM: 784 coefficients
  - Such redundancy makes computational cost demanding
    - e.g.) Model updating task using ROM requires the gradient of each coefficient

- **FEM**: $\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{f}_{nl}(\mathbf{x}) = \mathbf{f}(t)$



load case #1
load case #2
load case #3
…

- Some nonlinear coefficients are sensitive to how the load cases are formulated[5]
  - The accuracy of ROMs is not consistent to the different load cases used for Galerkin projection
  - Weakens the robustness of the ROMs

- **ROM**: $\ddot{q}_r + 2\zeta\omega_n\dot{q}_r + \omega_r^2\, q_r + \theta_r(q_1, q_2, \dots, q_n) = \boldsymbol{\phi}_r^T f(t)$

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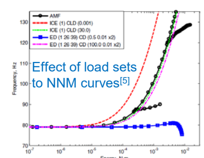

[5] R. J. Kuether, B. J. Deaner, J. J. Hollkamp, M. S. Allen, AIAA Journal 53 (11), 2015

5

A Data-Driven ROM

- This study presents a regression-based, data-driven ROM methodology for geometrically nonlinear structures
  - Issue #1) A Typical ROM only represents a single FEM configuration…
  - Issue #2) A Typical ROM is composed of redundant & sensitive nonlinear coefficients…

6

A Gaussian Process Regression Reduced Order Model of GNL Structures - Presenter : Kyusic Park | Introduction

A Data-Driven ROM

- This study presents a regression-based, data-driven ROM methodology for geometrically nonlinear structures

  - ~~Issue #1) A Typical ROM only represents a single FEM configuration~~
  - The proposed ROM is a trained **regression model**, which incorporates the parameter variations in FEM
  - Physical intuition can be transferred from FEM to ROM
    - e.g.) Sensitivity/gradient of ROM w.r.t. FEM
  - Eliminates a need of repetitive ROM computation, and directly predicts a ROM from any new FEM configuration

  - Issue #2) A Typical ROM is composed of redundant & sensitive nonlinear coefficients

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A Gaussian Process Regression Reduced Order Model of GNL Structures - Presenter : Kyusic Park | Introduction

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  - Eliminates a need of repetitive ROM computation, and directly predicts a ROM from any new FEM configuration

  - ~~Issue #2) A Typical ROM is composed of redundant & sensitive nonlinear coefficients~~
  - The proposed ROM applies **Gaussian Process Regression (GPR),** and evaluates the uncertainty of ROM parameters with respect to the applied loads for a given set of training data
  - Enhances the robustness and computational efficiency of ROM by filtering out the sensitive nonlinear ROM parameters

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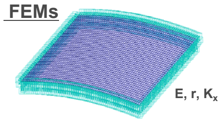
Gaussian Process Regression ROM (GPR ROM)

- Gaussian Process Regression (GPR) is applied to the ROM
  - A powerful tool that specifies any finite collection of random variables by its mean and covariance function based on the assumption that the collection obeys a joint Gaussian distribution[6]
  - Non-parametric: only need to specify mean and covariance functions that are suitable for the problem
  - Flexible: can be either expressed by a simple or a complicated manner depending on the complexity of the problem
    …
  - Model prediction confidence:
    - Quantify the uncertainty of model prediction with respect to the input variables

[6] Carl Edward Rasmussen, lectures on machine learning. Gaussian Processes in Machine Learning, pages 63–71, 2004.

GPR ROM Formulation

FEMs



$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{f}_{nl}(\mathbf{x}) = \mathbf{f}(t)$

ICE ROMs

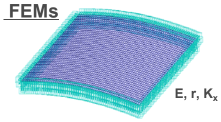
$$\ddot{q}_r + 2\zeta\omega_n\dot{q}_r + \omega_r^2\, q_r + \theta_r(q_1, \dots, q_n) = \boldsymbol{\phi}_r^T f(t)$$

$$\theta_r = \sum_{i=1}^{m}\sum_{j=i}^{m} \alpha_r(i,j)\, q_i q_j + \sum_{i=1}^{m}\sum_{j=i}^{m}\sum_{k=j}^{m} \beta_r(i,j,k)\, q_i q_j q_k$$

- Training sets for GPR ROM
  - Input training sets: FEMs with varying parameters $\mathbf{P}_{\mathrm{tr}} = [\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_{N_{\mathrm{tr}}}]^{\mathrm{T}}$
    - Number of training sets: N_{tr}
    - Number of features: d
  - Outputs training sets: ICE ROM coefficient, $\mathbf{y}_{\mathrm{tr}} = [\mathrm{y}_1, \mathrm{y}_2, \dots, \mathrm{y}_{N_{\mathrm{tr}}}]^{\mathrm{T}}$

GPR ROM Formulation (cont'd)

FEMs



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- Training sets for GPR ROM:
  - FEMs parameters $\mathbf{P}_{\mathrm{tr}} = [\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_{N_{\mathrm{tr}}}]^{\mathrm{T}}$
    - Number of training sets: N_{tr}
    - Number of features: d
  - ROM coefficient, $\mathbf{y}_{\mathrm{tr}} = [y_1, y_2, \dots, y_{N_{\mathrm{tr}}}]^{\mathrm{T}}$

1. GP model of the ROM coefficient, y
   - A GP function ψ describes the distribution of ROM coefficient y, which can be written as:

     $\psi(\mathbf{p}) \sim GP(\mu, \kappa)$ (μ: function of mean, κ: function of covariance matrix)

     where the covariance function κ is defined with a kernel *(ARD SE kernel in this work)*

     $$\kappa(\mathbf{p}_i, \mathbf{p}_j) = \sigma_f^2 \exp\left(-\frac{1}{2}\sum_{m=1}^{d}\frac{(\mathbf{p}_{i,m} - \mathbf{p}_{j,m})^2}{\sigma_m^2}\right)$$

   - The distribution can be corrupted by any independent noise term ϵ_ψ, so the final GP model y can be expressed as,

     $$y(\mathbf{p}) = \psi(\mathbf{p}) + \epsilon\psi, \quad \epsilon_\psi \sim GP(0, \sigma_n^2),$$

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GPR ROM Formulation (cont'd)

2. Training of GPR ROM
   - Trained by optimizing the hyperparameters of its mean and covariance functions $\boldsymbol{\theta}_h = \{\sigma_f, \sigma_1, \sigma_2, \dots, \sigma_d, \sigma_n\}$
   - This is achieved by estimating the maximum likelihood (MLE) of the GP model with respect to the input training set:

$$\boldsymbol{\theta}_h = \underset{\boldsymbol{\theta}_h}{\operatorname{argmax}} \log p\,(\mathbf{y}_{\mathrm{tr}}|\mathbf{P}_{\mathrm{tr}}) = \underset{\boldsymbol{\theta}_h}{\operatorname{argmax}}\left\{-\frac{1}{2}\log|\boldsymbol{\Sigma}(\boldsymbol{\theta}_h)| - \frac{1}{2}(\mathbf{y}_{\mathrm{tr}} - \boldsymbol{\mu}(\boldsymbol{\theta}_h))^{\mathrm{T}}\boldsymbol{\Sigma}_y^{-1}(\boldsymbol{\theta}_h)(\mathbf{y}_{\mathrm{tr}} - \boldsymbol{\mu}(\boldsymbol{\theta}_h)) - \frac{N_{\mathrm{tr}}}{2}\log(2\pi)\right\}$$

3. Prediction using the GPR ROM
   - The distribution of the training sets can be used as a prior from a Bayesian perspective
   - The posterior distribution (a new test set $\mathbf{y}_*$) can be estimated using the conditional distribution of joint Gaussian variables

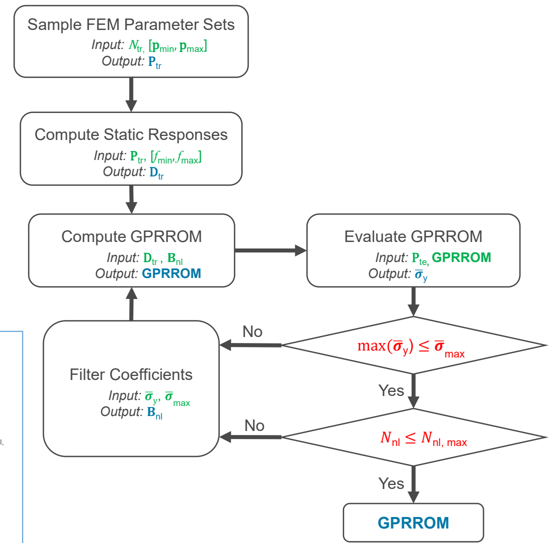
$$\mathbf{y}_*|\mathbf{P}^*, \mathbf{P}_{\mathrm{tr}}, \mathbf{y}_{\mathrm{tr}} \sim \mathcal{N}(\,\boldsymbol{\mu}_* + \boldsymbol{\Sigma}_*^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{y}_{\mathrm{tr}} - \boldsymbol{\mu})\,,\ \boldsymbol{\Sigma}_{**} - \boldsymbol{\Sigma}_*^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_*\,)$$

$\boldsymbol{\Sigma}_*$: training-test set covariance $\kappa(\mathbf{P}_{\mathrm{tr}}, \mathbf{P}^*)$, $\boldsymbol{\Sigma}_{**}$: test set covariance $\kappa(\mathbf{P}^*, \mathbf{P}^*)$

   - The prediction can be estimated by the mean $\boldsymbol{\mu}_* + \boldsymbol{\Sigma}_*^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{y}_{\mathrm{tr}} - \boldsymbol{\mu})$ and its variance by $\boldsymbol{\Sigma}_{**} - \boldsymbol{\Sigma}_*^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_*$
     - "Quantifies the uncertainty of model prediction"

GPR ROM Training Framework

- GPR ROM captures the variability of FEM parameters and quickly estimates a ROM and its predictive confidence for a new input FEM
- The framework utilizes the predictive uncertainty of the GPR ROM to define the optimal coefficient set
  - Predictive uncertainty with respect to the input static load cases in Galerkin projection
- Applies random forcing f_r for computing the ROM
  - If GPR ROM coefficient is certain to f_r, → keep in the ROM
  - If GPR ROM coefficient is uncertain to f_r → discard from the ROM (that coefficient is sensitive, so it weakens the robustness of the ROM)



- GPR ROM framework parameters:
  - Number of FEM training sets, N_{tr}
  - Upper & lower bounds of FEM parameters, $[\mathbf{p}_{min}, \mathbf{p}_{max}]$
  - Random force scaling bound, $[f_{min}, f_{max}]$
  - Maximum allowable prediction STD per mode, $\bar{\sigma}_{max}$
  - Maximum allowable number of nonlinear ROM coefficient, $N_{nl,\,max}$
  - Number of FEM testsets, N_{te}
- Output: **GPR ROM**
  - ROM training coefficients, $\mathbf{P}_{tr}$
  - Optimized hyperparameters of mean & covariance function, $\boldsymbol{\theta}_h$
  - Boolean matrix of nonlinear coefficients, $\mathbf{B}_{nl}$

1. Flat Beam

- Model parameters
  - Length: 228.6 mm
  - Width: 12.7 mm (0.5 in)
  - Thickness: 0.7874 mm (0.031 in)
  - Density: 7,870 kg/m^3 (7.36 x 10^{-4} lb-s^2/in^4)
  - Young's modulus (E): 2.07 x 10^5 N/mm^2 (2.97 x 10^7 lb/in^2)
  - Axial spring stiffness (K_x) : 1.0 x 10^4 to 5.0 x 10^5 (lbf/in)
  - 40 beam elements

- Input
  - Boundary stiffness (K_x) approximated by axial springs
    - One of the most uncertain parameters in any practical application
    - Significantly impacts the nonlinear coefficients of ROMs
      - ** K_x has no impact on the linear frequencies
  - Uniformly sampled within the range of 1 X10^4 to 5 X10^5 (lbf/in)
    - The bounds are chosen to cover from soft to stiff (near to clamped) condition
- Output: GPR-ROM
  - 2-DOF GPR ROM (Mode 1 & 3)

1.1. 2DOF GPR ROM of Flat Beam

- Number of training samples, N_{tr} = 20
  - Total number of load cases: 20 x 7 = 140
- Random load scaling factor, $f_r \in [0.25, 0.75]$ x beam thickness
- Number of testsets for prediction, N_{tr} = 900

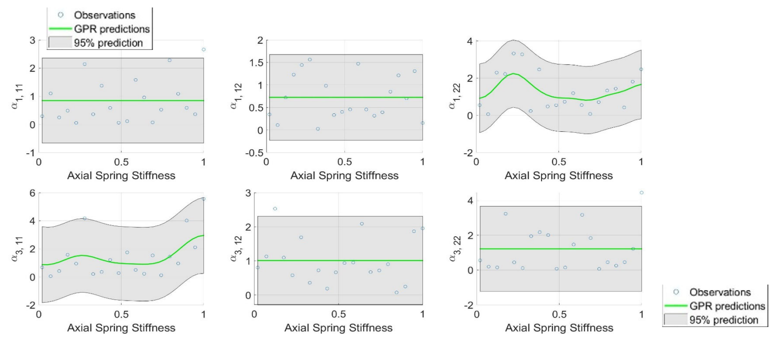
- The predicted quadratic terms were very uncertain and had large variances ($\bar{\sigma}_{Knl}$) from being zero
  - They were very sensitive to the load scaling factor, f_r
  - Corresponds with the known physics: quadratic terms have negligible effect on the symmetric beams

<STD of the GPR ROM predicted by the test sets, Mode 1 >

	$\alpha_{1,11}$	$\alpha_{1,33}$	$\alpha_{1,13}$	$\beta_{1,111}$	$\beta_{1,113}$	$\beta_{1,331}$	$\beta_{1,333}$
$\bar{\sigma}_y$	0.770	0.924	0.485	0.003	0.008	0.003	0.003

<STD of the GPR ROM predicted by the test sets, Mode 3 >

	$\alpha_{3,11}$	$\alpha_{3,33}$	$\alpha_{3,13}$	$\beta_{3,111}$	$\beta_{3,113}$	$\beta_{3,331}$	$\beta_{3,333}$
$\bar{\sigma}_y$	1.346	1.247	0.663	0.003	0.032	0.012	0.005



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- The predicted cubic terms were very certain and gradually captured from soft to clamped boundary condition

<STD of the GPR ROM predicted by the test sets, Mode 1 >

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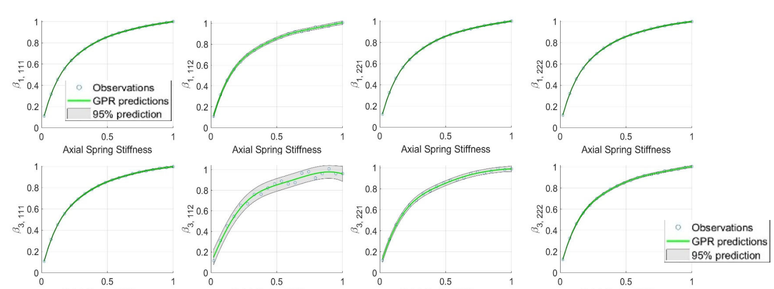

1.2. Coefficient Filtering of GPR ROM

- Redundant (uncertain) coefficients were filtered using mean STD threshold of each mode, ($\bar{\boldsymbol{\sigma}}_{max} = [\bar{\sigma}_{tmax,\ Mode\ 1}, \bar{\sigma}_{max,\ Mode\ 2}] = [0.07, 0.07]$)
- The quadratic terms were all removed by the filtering algorithm (N_{nl} = 14 → 8)
- The mean STD of the GPR ROM significantly decreased: $\bar{\boldsymbol{\sigma}}_{GPR}$= 0.393 → 0.009
- The confidence intervals ($\bar{\sigma}_y$) of the cubic terms remained the same
- ROM became small but robust with respect to the force scaling factor

<STD of the GPR ROM predicted by the test sets, Mode 1 >

	~~$\alpha_{1,11}$~~	~~$\alpha_{1,33}$~~	~~$\alpha_{1,13}$~~	$\beta_{1,111}$	$\beta_{1,113}$	$\beta_{1,331}$	$\beta_{1,333}$
$\bar{\sigma}_y$				0.003	0.008	0.003	0.003

<STD of the GPR ROM predicted by the test sets, Mode 3 >

	~~$\alpha_{3,11}$~~	~~$\alpha_{3,33}$~~	~~$\alpha_{3,13}$~~	$\beta_{3,111}$	$\beta_{3,113}$	$\beta_{3,331}$	$\beta_{3,333}$
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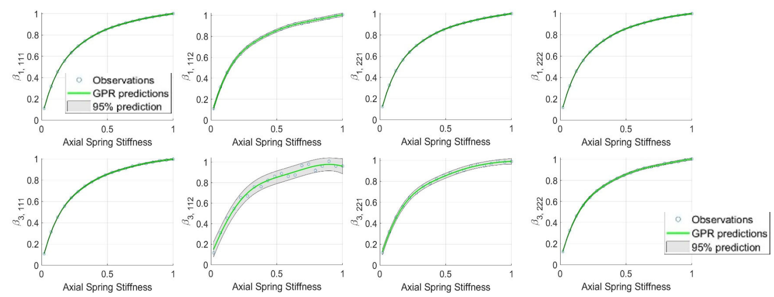

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1.2. Coefficient Filtering of GPR ROM (cont'd)

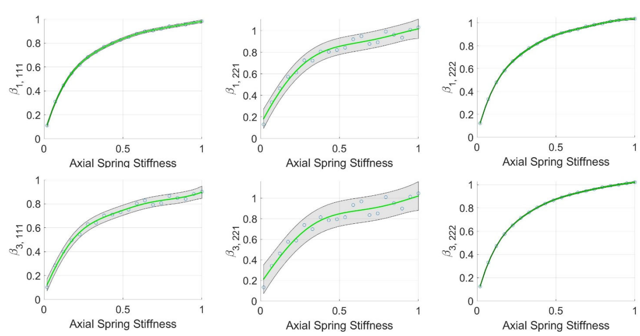
- N_{nl} = 14 → 8 → 6
- The mean STD of the GPR ROM slightly increased ($\bar{\boldsymbol{\sigma}}_{GPR}$= 0.393 → 0.009 → 0.023)
  - The variances $\bar{\sigma}_y$ of the cubic terms slightly increased

<STD of the GPR ROM predicted by the test sets, Mode 1 >

	~~$\alpha_{1,11}$~~	~~$\alpha_{1,33}$~~	~~$\alpha_{1,13}$~~	$\beta_{1,111}$	~~$\beta_{1,113}$~~	~~$\beta_{1,331}$~~	$\beta_{1,333}$
$\bar{\sigma}_y$				0.005		0.040	0.004

<STD of the GPR ROM predicted by the test sets, Mode 3 >

	~~$\alpha_{3,11}$~~	~~$\alpha_{3,33}$~~	~~$\alpha_{3,13}$~~	$\beta_{3,111}$	~~$\beta_{3,113}$~~	~~$\beta_{3,331}$~~	$\beta_{3,333}$
$\bar{\sigma}_y$				0.022		0.063	0.003



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1.2. Coefficient Filtering of GPR ROM (cont'd)

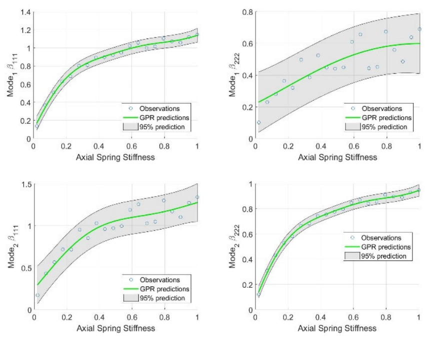
- N_{nl} = 14 → 8 → 6 → 4
- $\bar{\sigma}_{\text{GPR}}$ = 0.3931 → 0.0086 → 0.0230 → 0.0625
- The variance $\bar{\sigma}_y$ of the cubic terms became even more significant

<STD of the GPR ROM predicted by the test sets, Mode 1 >

	~~$\alpha_{1,11}$~~	~~$\alpha_{1,33}$~~	~~$\alpha_{1,13}$~~	$\beta_{1,111}$	~~$\beta_{1,113}$~~	~~$\beta_{1,331}$~~	$\beta_{1,333}$
$\bar{\sigma}_y$				0.034			0.091

<STD of the GPR ROM predicted by the test sets, Mode 3 >

	~~$\alpha_{3,11}$~~	~~$\alpha_{3,33}$~~	~~$\alpha_{3,13}$~~	$\beta_{3,111}$	~~$\beta_{3,113}$~~	~~$\beta_{3,331}$~~	$\beta_{3,333}$
$\bar{\sigma}_y$				0.104			0.021



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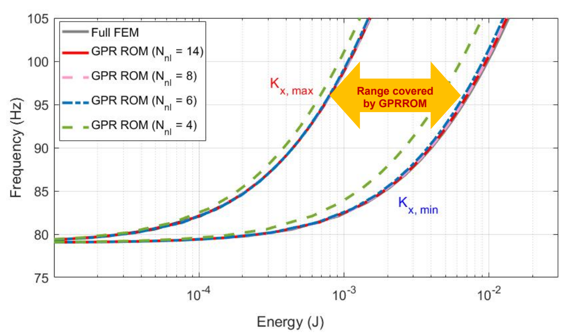
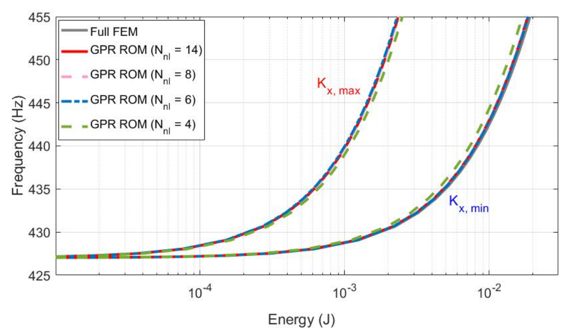
A Gaussian Process Regression Reduced Order Model of GNL Structures - Presenter : Kyusic Park | Numerical studies

1.3. NNM Curves of GPR ROMs

- The GPR ROM could predict the dynamic responses for a wide range of varying axial spring stiffness
- The accuracy broke down from N_{nl} = 4
  - Still, the accuracy was reasonable
  - This is because the flat beam structures do not entail a noticeable modal coupling





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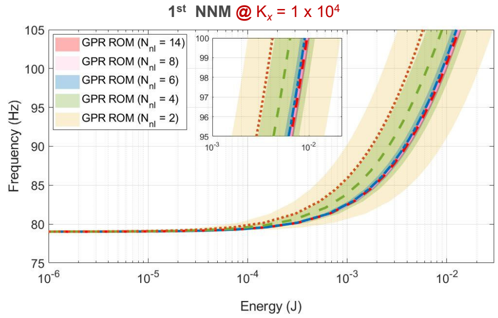
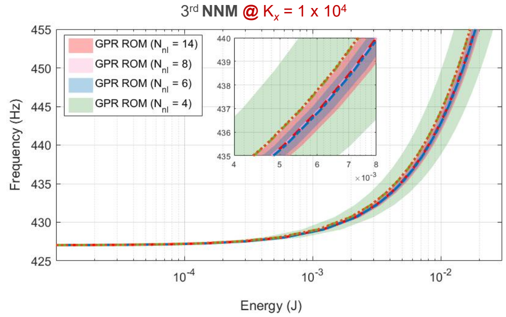
1.3. NNM Curves of GPR ROMs (cont'd)

- However, the prediction confidence dramatically reduced from N_{nl} = 4
  - This is obvious when the boundary is less stiff ($K_x = 1 \times 10^4$)
  - Trade off exists between the prediction confidence and reducing the GPR ROM coefficient set
  - Can be a powerful Indicator for choosing the optimal coefficient sets

1st NNM @ $K_x = 1 \times 10^4$



3rd NNM @ $K_x = 1 \times 10^4$



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2. Curved Beam

- Model parameters
  - Length: 180mm
  - Width: 8.32mm
  - Thickness: 2.6mm
  - Radius of curvature: 3,175mm
  - Young's modulus: 3.10×10^9 N/m^2
  - Density: 1,248.36 kg/m^3
  - Poisson's ratio: 0.33
  - 60 beam elements

- Inputs
  - Young's modulus (E) and radius of curvature (r)
    - The parameters have a significant uncertainties due to the manufacturing variability
    - They also have a significant effect on the dynamic response
  - Uniformly sampled within the range of $\pm 50\%$ of the nominal values
- Output: GPR-ROM
  - 3-DOF GPR ROM (Mode 1 & 2 & 3)

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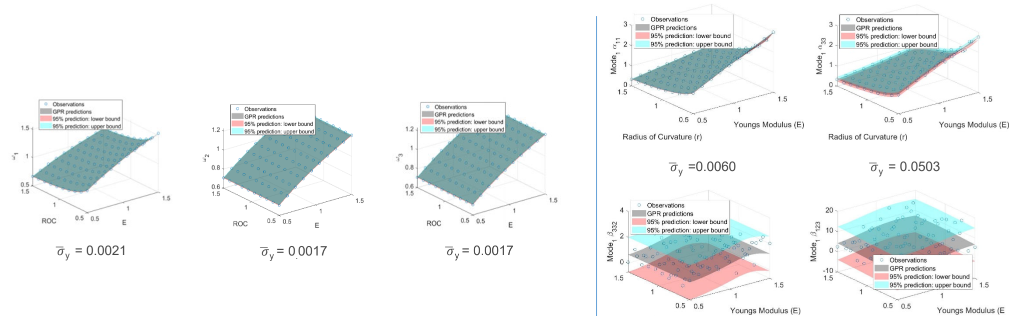
2.1. 3DOF GPR ROM of Curved Beam

- N_{tr} = 10 x 10 = 100
  - Total number of load cases: 100 x 16 = 1,600
- $f_r \in [0.25, 3.00]$ x beam thickness
- N_{te} = 900
- $\bar{\sigma}_{max} = [0.10,\ 0.10,\ 0.30]$

- The linear and nonlinear coefficients were sensitive to Young's modulus (E) than radius of curvature (r)
- Linear frequencies were very accurately predicted (almost no variance) by GPR-ROM
- There were some nonlinear terms having large prediction variance and thus could be filtered out
  - They were mostly non-resonant terms [7] , which contributes less to capturing the system's nonlinearity



$\bar{\sigma}_y$ = 0.0021 $\bar{\sigma}_y$ = 0.0017 $\bar{\sigma}_y$ = 0.0017

$\bar{\sigma}_y$ =0.0060 $\bar{\sigma}_y$ = 0.0503

$\bar{\sigma}_y$ = 0.6771 $\bar{\sigma}_y$ = 4.1176

[7] Shen, Yichang, et al., European Journal of Mechanics-A/Solids 86 (2021): 104165

2.2. GPR ROM Filtering for Redundant Coefficients

- N_{tr} = 10 x 10 = 100
  - Total number of load cases: 100 x 16 = 1,600
- $f_r \in [0.25, 3.00]$ x beam thickness
- N_{te} = 900
- $\bar{\sigma}_{max} = [0.10,\ 0.10,\ 0.30]$

- The nonlinear coefficients could be reduced by less than 50% by the filtering (N_{nl} = 48 → 21)
- The mean STD of the GPR ROM significantly decreased: $\bar{\sigma}_{GPR}$= 0.232 → 0.060

	Iteration #1	Iteration #2	Iteration #3	Iteration #4	Iteration #5	Iteration #6	Iteration #7
N_{nl}	48	38	29	24	23	21	**21**
$N_{nl,\ Mode\ 1}$ (max $\bar{\sigma}_y$)	16 (4.1176)	11 (0.6116)	7 (0.0879)	~~6 (0.2151)~~	7 (0.0879)	7 (0.0879)	**7 (0.0879)**
$N_{nl,\ Mode\ 2}$ (max $\bar{\sigma}_y$)	16 (0.5863)	14 (0.2612)	11 (0.3664)	8 (0.5014)	7 (0.0496)	~~6 (0.3871)~~	**7 (0.0496)**
$N_{nl,\ Mode\ 3}$ (max $\bar{\sigma}_y$)	16 (1.5380)	13 (0.3408)	11 (0.1897)	10 (0.5009)	9 (0.3919)	8 (0.2702)	**7 (0.1784)**
$\bar{\sigma}_{GPR}$	0.2323	0.1008	0.0713	0.1141	0.0803	0.0837	**0.0601**

2.3. Comparison between GPR ROMs computed by different force scaling bounds

- Larger load scaling bounds resulted in a larger predictive confidence of each coefficient
- The coefficients kept in the ROM were mostly the same between the cases with different load scaling bounds

Mode 1

f_r bounds	α_{11}	α_{22}	α_{33}	α_{12}	α_{13}	α_{23}	β_{111}	β_{112}	β_{113}	β_{221}	β_{222}	β_{223}	β_{331}	β_{332}	β_{333}	β_{123}	N_{nl}
1. $f_r \in$ [0.25, 0.75]	0.0161	0.0091	0.0082				0.0168			0.0132			0.0184		0.0532		#7
2. $f_r \in$ [0.25, 1.50]	0.0173	0.0059	0.0388				0.0149			0.0057		0.0561	0.0595		0.0396		#8
3. $f_r \in$ [0.25, 2.00]	0.0188	0.0059	0.0454				0.0204			0.0074		0.0627	0.0658		0.0475		#8
4. $f_r \in$ [0.25, 3.00]	0.0271	0.0188	0.0834				0.0268			0.0211			0.0879		0.0597		#7
5. $f_r \in$ [0.10, 3.00]	0.0176	0.0182	0.0743				0.0180			0.0208			0.0745		0.0455		#7

Mode 2

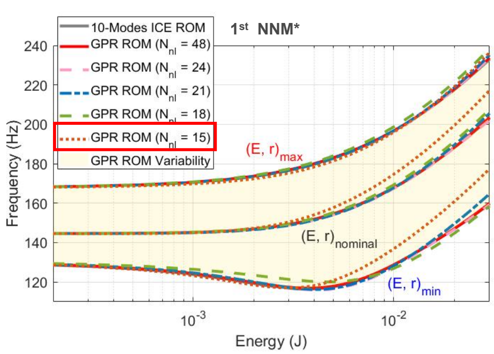
	α_{11}	α_{22}	α_{33}	α_{12}	α_{13}	α_{23}	β_{111}	β_{112}	β_{113}	β_{221}	β_{222}	β_{223}	β_{331}	β_{332}	β_{333}	β_{123}	N_{nl}
1. $f_r \in$ [0.25, 0.75]		0.0154	0.0127	0.0058		0.0098		0.0067			0.0038			0.0032		0.0059	#8
2. $f_r \in$ [0.25, 1.50]		0.0155	0.0201	0.0088		0.0311		0.0137			0.0092			0.0046		0.0158	#8
3. $f_r \in$ [0.25, 2.00]		0.0196	0.0244	0.0132		0.0333		0.0181			0.0102			0.0062		0.0214	#8
4. $f_r \in$ [0.25, 3.00]		0.0315		0.0219		0.0357		0.0256			0.0147			0.0206		0.0496	#7
5. $f_r \in$ [0.10, 3.00]		0.0222		0.0188		0.0319		0.0236			0.0130			0.0196		0.0868	#7

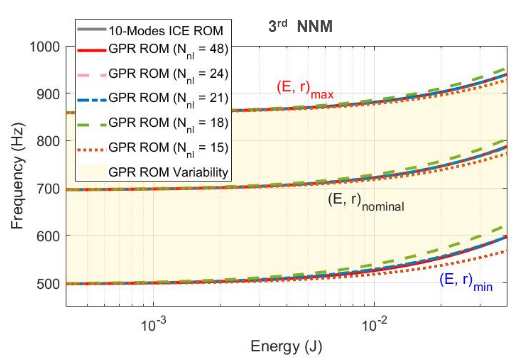
Mode 3

	α_{11}	α_{22}	α_{33}	α_{12}	α_{13}	α_{23}	β_{111}	β_{112}	β_{113}	β_{221}	β_{222}	β_{223}	β_{331}	β_{332}	β_{333}	β_{123}	N_{nl}
1. $f_r \in$ [0.25, 0.75]		0.0113	0.0445							0.0512		0.0092	0.0412		0.0244		#6
2. $f_r \in$ [0.25, 1.50]		0.0669	0.1083							0.0980		0.0316	0.0506		0.0216		#6
3. $f_r \in$ [0.25, 2.00]		0.0712	0.1301			0.1174				0.1023		0.0546	0.0618		0.0256		#7
4. $f_r \in$ [0.25, 3.00]		0.1106	0.1784			0.1301				0.1274		0.0769	0.0824		0.0322		#7
5. $f_r \in$ [0.10, 3.00]		0.1044	0.1644			0.1060				0.1057		0.0676	0.0834		0.0243		#7

2.4. NNM Curves of GPR ROMs

- The GPR ROM accurately predicted the dynamic responses for a wide range of varying FEM parameters
  - Snap-through instability was also well captured
- The accuracy was maintained when reducing more than 50% of the number of coefficients
  - Accuracy broke down from N_{nl} = 15 (33%)



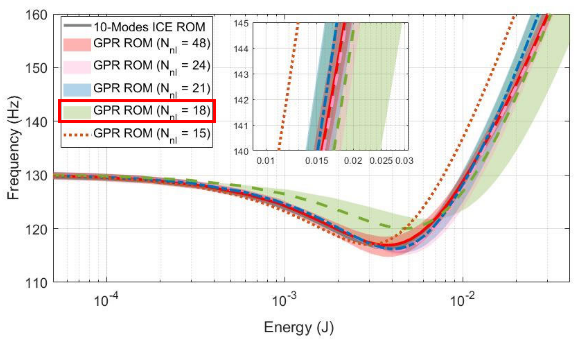


(*NNM computed by the GPR ROM with $f_r \in$ [0.25, 3.00] x beam thickness)

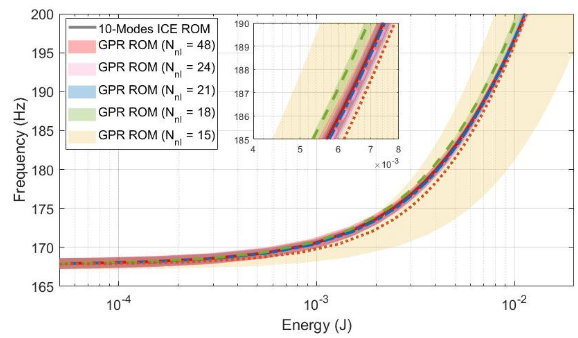
2.4. NNM Curves of GPR ROMs (cont'd)

- Prediction confidence dramatically reduced from N_{nl} = 18
  - Trade off exists between the prediction confidence and reducing the GPR ROM coefficient set
  - Highlights the merit of using the GPR ROMs to predict confidence intervals of nonlinear responses

1st NNM @ min E, min r



1st NNM @ max E, max r



2.5. Computational Efficiency

- Offline stage: a considerable number of samples need to be trained
  - Parallel computing is applicable to compute the static load displacement data
- Online stage: no computational effort is needed to create a ROM for any new input FEM parameters
- Computational cost
  - Ex) 3-DOF GPR ROM of curved beam model using 100 training FEM sets

	Offline Training (s)	**Online computation for a ROM** (s)	**Time integration** (s) (T = 10s, sample rate = 10,000)
GPR ROM	501.60	0.01	8.86
ICE ROM	-	4.93	8.87
FEM (62 elements)	-	-	5652. 84

* used Intel Core i7-7700K 4.2GHz quad-core computer with 64 GB of RAM

Conclusion

- Proposed data-driven ROM that takes advantage of known physics of the problem and based on a form that is known to be efficient and accurate
- GPR ROM accurately captures the variations of FEM parameters with an optimally reduced set of nonlinear coefficients
- GPR ROM can compute the variability in the responses that are due to the uncertainty of FEM design parameters.
- Computational efficiency can be greatly enhanced
  - ROM coefficients can be pre-computed with high confidence with small and sparse set of training data (offline stage)
  - ROM can be directly produced for any new input FEM (no need of static analysis during online stage)

- Currently exploring GPR ROM application to Nonlinear Model updating
  - ROM can accelerate the model updating tasks, but it is challenging to correlate FEM (physical domain) to the updated ROM[8]
  - FEM variation and uncertainty are directly linked to GPR ROM
    - Sensitivity of ROM coefficient w.r.t. FEM parameter, $\frac{\partial \alpha}{\partial P}$, can be analytically derived without a need of numerical effort (e.g. finite difference)
  - No need to iterate Galerkin projection when correlating FEM to ROM

- Potential application to Thermal model updating using Gaussian Process Regression Model
  - Significant uncertainty has been explored when GNL structures are subjected to thermal stress[9]

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[8] C. I. Van Damme, M. S. Allen, J. J. Hollkamp, AIAA Journal (2020) 1–16.
[9] K. Park, and M. S. Allen., Proceedings of IMAC 39th, 83-93. 2021

Thank you!

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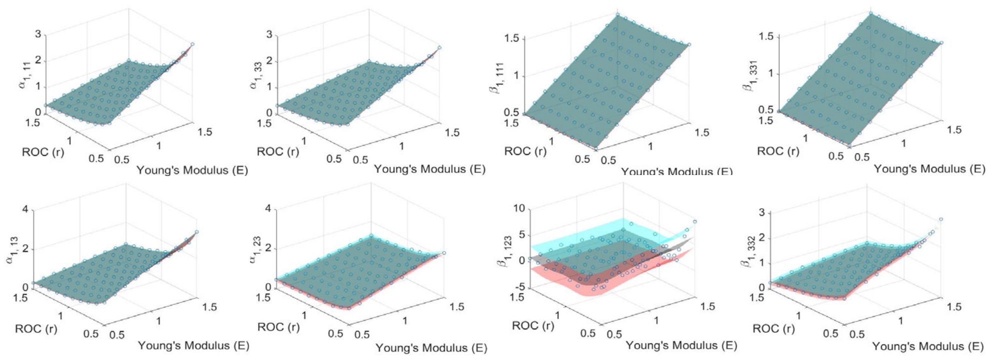
Appendix A. Effect of Random Load Scaling Factor

- N_{tr} = 10 x 10 = 100
  - Total number of load cases: 100 x 16 = 1,600
- $f_r \in [0.25, 0.75]$ x beam thickness
- N_{te} = 900
- $\bar{\sigma}_{max} = [0.08,\ 0.08,\ 0.08]$

- Reduced bounds of force scaling (f_r) result in the decreased confidence variance for each coefficient



Appendix A. Effect of Random Load Scaling Factor (cont'd)

- N_{tr} = 10 x 10 = 100
  - Total number of load cases: 100 x 16 = 1,600
- $f_r \in [0.25, 0.75]$ x beam thickness
- N_{te} = 900
- $\bar{\sigma}_{max} = [0.08,\ 0.08,\ 0.08]$

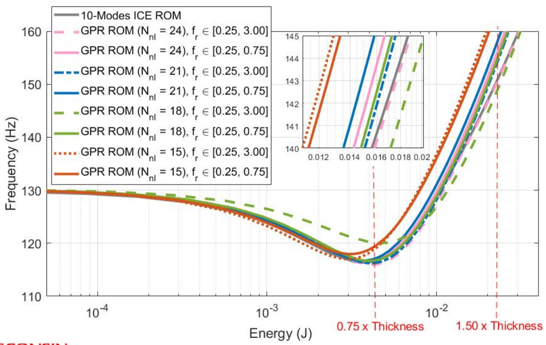
- Reduced bounds of f_r result in the decrease of $\bar{\sigma}_{GPR}$ and $\bar{\sigma}_y$ of each coefficients
- The optimal (filtered) coefficients of each mode are similar to the previous case

	Iteration #1	Iteration #2	Iteration #3	Iteration #4	Iteration #5	Iteration #6	Iteration #7	Iteration #8	Iteration #9	Final
N_{nl}	48	42	38	35	30	25	22	20	20	21
$N_{nl,\ Mode\ 1}$ (max $\bar{\sigma}_y$)	16 (0.8375)	14 (0.1623)	13 (0.0512)	12 (0.0933)	11 (0.0266)	10 (0.0151)	9 (0.1885)	7 (0.0532)	~~6 (0.1147)~~	7 (0.0532)
$N_{nl,\ Mode\ 2}$ (max $\bar{\sigma}_y$)	16 (0.0771)	15 (0.0437)	14 (0.0689)	13 (0.1411)	11 (0.1195)	9 (0.3457)	8 (0.0154)	~~7 (0.1384)~~	8 (0.0154)	8 (0.0154)
$N_{nl,\ Mode\ 3}$ (max $\bar{\sigma}_y$)	16 (0.3141)	13 (0.1813)	11 (0.0153)	10 (0.1199)	8 (0.2565)	6 (0.0512)	~~5 (0.1412)~~	6 (0.0512)	6 (0.0512)	6 (0.0512)
$\bar{\sigma}_{GPR}$	0.0425	0.0204	0.0116	0.0279	0.0308	0.0262	0.0335	0.0292	0.0323	0.0181

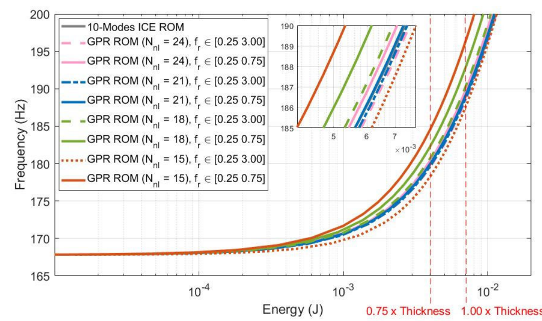
Appendix B. NNM Curves of GPR ROMs: effect of load scaling

- When load scaling bounds decrease from [0.25, 3.00] to [0.25, 0.75],
  - The accuracy is maintained when reducing more than 50% of the number of coefficients
    - Accuracy breaks down from N_{nl} = 15 (33%)
  - Better prediction in smaller load (energy) level, including snap-through (e.g. N_{nl} = 18)
  - Accuracy generally decreases for the large amplitude → less capable of capturing the large amplitude due to small loading range
  - Better to use larger load scaling to capture the large deformations

1st NNM @ min E, min r



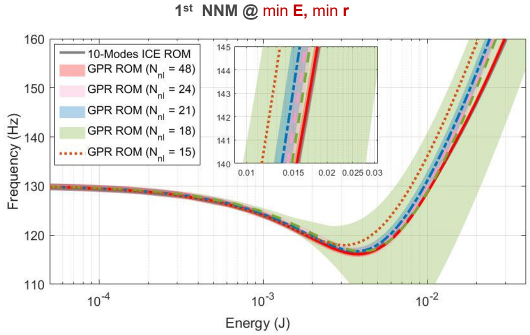
1st NNM @ max E, max r



Appendix B. NNM Curves of GPR ROMs: effect of load scaling (cont'd)

- When load scaling bounds decrease from [0.25, 3.00] to [0.25, 0.75],
  - The uncertainty of GPR ROM decreased but it does not result in better predictive confidence of NNMs
  - Shows even larger confidence interval at large displacements (e.g. N_{nl} = 18)

1st NNM @ min E, min r



1st NNM @ max E, max r