# NIXO-Based Identification of the Dominant Terms in a Nonlinear Equation of Motion

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**Abstract**

While many algorithms have been proposed to identify nonlinear dynamic systems, nearly all methods require that the form of equation of motion is known a priori. Examples of very effective methods of this kind are NIFO, CRP and NARMAX. Several works have sought to extend NARX or NARMAX to a black-box modeling technique. They have proven to be successful in finding accurate mathematical models for certain types of nonlinear systems yet no method has proved universally successful. This work presents and evaluates a new black-box identification approach based on a new NIFO/CRP type algorithm called Nonlinear Identification through eXtended Outputs (NIXO). The proposed algorithm expresses the nonlinear part of equation of motion as a polynomial of high order and then removes the terms which are classified (with high probability) as irrelevant in the mechanical system's response. This division into dominant and irrelevant nonlinear terms relies on the values of two novel indicators that are particular to NIXO. This technique is demonstrated on a numerical case study employing a curved beam. Then the method will be used to estimate the NLEOM of flat and curved beams that were manufactured using a 3D printer. The experimental results will be validated against those obtained using phase resonance testing, which identifies a nonlinear normal mode (NNM) of the system using a vastly different approach.

**Keywords**: Nonlinear system identification, Nonlinear parameter estimation, Black-box methods, Nonlinear Normal Modes, NIXO methods

**Definition of the Δ-indicators**

The objective of this publication is to propose a new technique for black-box nonlinear system identification. This work builds on that presented in [1] where the authors introduced a new frequency-domain system ID algorithms called $\mathbf{D}_1$- and $\mathbf{D}_2$-NIXO, for Nonlinear Identification through eXtended Outputs. The NIXO base formulas are proposed in two versions. The first one estimates the nonlinear coefficients as complex numbers, while the second enforces them to be found as real. The reader can refer to [1] for a detailed derivation of these expressions.

While testing NIXO in several case studies, the authors noticed an interesting feature of the algorithm. Namely, it was observed that if the tested structure: (*i*) is excited with a swept (co)sine forcing signal and (*ii*) oscillates at low (but sufficiently high) amplitudes during the experiment, then the nonlinear terms that are not dominant in the system's response tend to be complex-valued. Hence, if we analyze the structure assuming the most general form of the nonlinear EOM – see e.g. Eq. (2) – then the NIXO algorithms will point out which nonlinear terms should be kept and which could be removed from the equation of motion. The complexity of the result can be quantified by two system identification metrics called $\Delta_*$ and $\Delta_{**}$, which are defined in Eq. (1).

$$\Delta_* = \left| \frac{\mathrm{Re}\left\{\beta_{cmplx}\right\} - \beta_{real}}{\beta_{real}} \right| \qquad \Delta_{**} = \frac{\left|\mathrm{Re}\left\{\beta_{cmplx}\right\}\right| - \left|\mathrm{Im}\left\{\beta_{cmplx}\right\}\right|}{\left|\mathrm{Re}\left\{\beta_{cmplx}\right\}\right|} \qquad \begin{cases} \Delta_* < \epsilon \\ \Delta_{**} > 1-\epsilon \end{cases} \tag{1}$$

where $\beta_{real}$ and $\beta_{cmplx}$ are the parameters identified by the complex and real versions of the NIXO algorithms, respectively.

As shown in Eq. (1), the $\Delta_*$-indicator expresses the relative difference between the coefficient that was enforced to be found as real and the real part of the complex one. Hence, $\Delta_*$ will be small when the two algorithms produce consistent results. In contrast, $\Delta_{**}$ is defined as the relative difference between the real and imaginary parts of the complex solution, and gives a measure of how large the imaginary part of the solution is. Hence, the nonlinear term is considered to be dominant when its $\Delta_*$ value is low enough and $\Delta_{**}$ is close to 1. Note that these two requirements must be satisfied simultaneously. The accuracy thresholds could be specified with a parameter $\epsilon$, as presented in Eq. (1c). The value of $\epsilon$ should be a small number, say $\epsilon = 0.05$. With the two Δ-indicators defined, we are ready to illustrate the black-box capabilities of the NIXO algorithms. The next sections present a successful black-box identification performed on a numerical model of a curved beam.

### Simulated Black-Box Identification of a Curved Beam

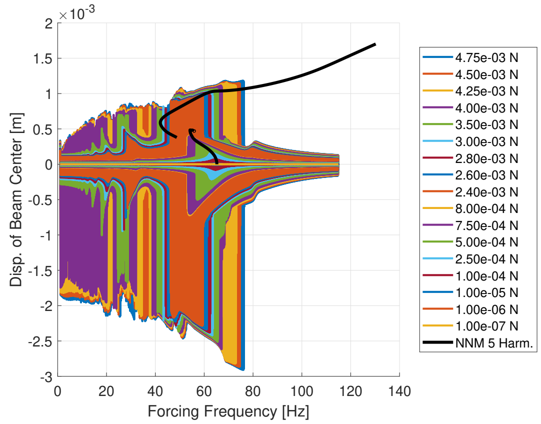


**Fig. 1:** The NNM curve is overlayed on the response at the beam's center to swept cosine input forces of varying amplitudes.

The numerical test is performed on an ICE-ROM of a clamped-clamped curved beam subjected to a uniformly distributed swept cosine forcing signal. The beam has a length of 304.8 mm, width of 12.7 mm, thickness of 0.508 mm and a radius of curvature of 11.43 m. It is made of steel with a Young's modulus of $207.4334 \times 10^{11}$ GPa, a density of 7850 kg/m$^3$ and a Poisson's ratio of 0.29.

The ICE-ROM consists of the first three symmetric modes, i.e. modes 1, 3 and 5. Their linear natural frequencies and damping ratios are: $\{65.181,\ 158.636,\ 385.882\}$ Hz and $\{0.035,\ 0.0262,\ 0.0174\}$, respectively. The nonlinear equation of motion of the system, including every possible nonlinear term is presented in (2). Since $(i)$ the nonlinear part consists of the quadratic and cubic parts, and $(ii)$ there are 3 modes present in the ROM – the number of nonlinear terms that can occur in the EOM is at most 16. In each case study run, we assume the most general form of the NLEOM, see Eq. (2), hence NIXO can point out the terms dominant in the system's response out of the most general set of 16 terms. This brief publication focuses on identification of the nonlinear mode 1, thus we presented the nonlinear equation of motion of this mode only; the equations for the two remaining modes are analogous. Note that the subscripts of the nonlinear coefficients correspond to the product of polynomial terms they multiply; e.g.: $\beta_{111}$ multiplies term $q_1^3$, while $\beta_{123}$ multiplies term $q_1 q_2 q_3$.

$$\underbrace{\ddot{q}_1 + 2\zeta_1\omega_1\dot{q}_1 + \omega_1^2 q_1}_{\text{linear part}} + \underbrace{\alpha_{11}^1 q_1^2 + \alpha_{12}^1 q_1 q_2 + \ldots}_{\text{quadratic stiffness part}} + \underbrace{\beta_{111}^1 q_1^3 + \beta_{112}^1 q_1^2 q_2 + \ldots}_{\text{cubic stiffness part}} = \mathbf{\Phi}_1^T \mathbf{f}(t) \tag{2}$$

### Identification of Mode 1

The beam is excited with swept cosine signals of various magnitudes, such that it oscillates at different response-levels in every test. These input/output signals are later provided to the NIXO algorithms, which uses them to estimate the underlying linear as well as the nonlinear parts of the system.

Since the 1st mode occurs at approximately 65 Hz, the authors decided to excite the system with 300-second-long (up and/or down) sweeps, with frequencies ranging from 1 to 115 Hz. The output signals obtained in these numerical tests are illustrated in Fig. 1 and can be grouped into two sets: down- and up-sweeps, which correspond to the force amplitudes of $F_0 \in \{1 \times 10^{-7},\ \ldots,\ 8 \times 10^{-4}\}$ and $F_0 \in \{2.4,\ \ldots,\ 4.75\} \times 10^{-3}$ newtons, respectively. Note that some of the responses shown are not symmetrical with respect to the equilibrium position. This result was expected, since the beam is curved and the *snap-through* effect causes this asymmetry. Moreover, this observation explains the importance of including the quadratic stiffness terms in the nonlinear equation of motion.

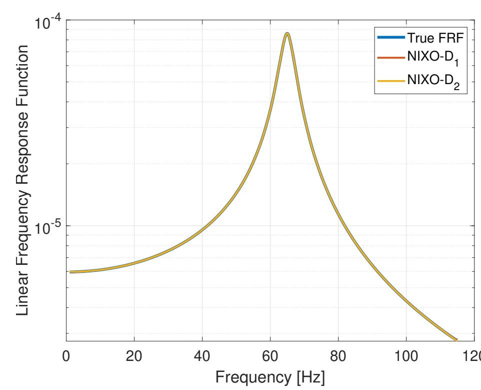


**Fig. 2:** Underlying linear system estimated successfully by the NIXO algorithms.

### Sample Black-Box ID Outcomes from High-Amplitude Vibration Tests

The results from a black-box system identification attempt are presented in this section. The input signals provided to the NIXO algorithms are the sweep cosines with magnitudes of $2.4 \times 10^{-3}$ (up-sweep) and $5.0 \times 10^{-4}$ newtons (down-sweep). The corresponding output signals were already shown in Fig. 1. Note that the signals used have very different amplitudes; the proposed algorithm seemed to work best when this is the case.

The outcomes from the case study are presented in Fig. 2 and Tab. 1. Figure 2 compares the true linear Frequency Response Function to those returned by NIXOs. The FRFs match perfectly, showing that the underlying linear system was successfully identified. Table 1 presents the four nonlinear terms (out of sixteen assumed beforehand) that were pointed out by NIXO algorithms as dominant in the system response (the coefficients which does not meet

**Tab. 1:** Estimated values of the nonlinear coefficients obtained using NIXO methods. Parameters marked with green satisfy the accuracy criteria. Parameters marked with blue are close to satisfying these requirements. The parameters that does not meet the accuracy criteria specified in Eq. (1c) are not shown in this table.

| | $\mathbf{D}_1$–NIXO | | | | | $\mathbf{D}_2$–NIXO | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $Re\{\cdot_{cmplx}\}$ | $Im\{\cdot_{cmplx}\}$ | $\cdot_{real}$ | $\Delta_*$ [%] | $\Delta_{**}$ [%] | $Re\{\cdot_{cmplx}\}$ | $Im\{\cdot_{cmplx}\}$ | $\cdot_{real}$ | $\Delta_*$ [%] | $\Delta_{**}$ [%] |
| $\alpha_{11}$ | 3.42E+09 | -4.33E+06 | 3.42E+09 | 0.16 | 99.87 | 3.34E+09 | -4.62E+07 | 3.44E+09 | 2.99 | 98.62 |
| $\alpha_{12}$ | 3.67E+09 | 2.38E+08 | 3.94E+09 | 6.89 | 93.51 | 4.29E+09 | 2.36E+08 | 3.73E+09 | 14.96 | 94.50 |
| $\beta_{111}$ | 2.04E+13 | 1.68E+11 | 2.01E+13 | 1.51 | 99.18 | 1.94E+13 | -4.71E+11 | 2.03E+13 | 4.71 | 97.57 |
| $\beta_{112}$ | 6.88E+13 | 5.48E+12 | 6.58E+13 | 4.65 | 92.04 | 6.47E+13 | -2.59E+12 | 6.28E+13 | 3.00 | 95.99 |

the accuracy criteria (1c) are not presented in the table). These terms correspond to the following parameters: $\alpha_{11}$, $\alpha_{12}$, $\beta_{111}$ and $\beta_{112}$. The nonlinear equation of motion including these four terms only does not loose much accuracy when compared to the true NLEOM containing the full set of 16 terms (for more details see the section below, where the results are validated). Since this is the smallest model that captures most of the structure's dynamics - these four nonlinear terms must be dominant in the mechanical system. Naturally, if more nonlinear coefficients has to be identified, then the signals from tests where the structure oscillates at higher amplitudes should be provided to NIXO. Then – besides estimating accurately the four parameters itemized above – the NIXO algorithms will additionally point out the nonlinear terms that are next on the importance list.

### Results Validation

To prove that the polynomial terms pointed out by NIXO are dominant in the system, we compare the true NNM curve with the one computed for the EOM with the reduced set of the nonlinear terms pointed out by NIXO, see Eq. (3). The values of coefficients $\alpha_{11}$, $\alpha_{12}$, $\beta_{111}$ and $\beta_{112}$ are presented in Tab. 1.

$$\ddot{q}_1 + 2\zeta_1\omega_1\dot{q}_1 + \omega_1^2 q_1 + \alpha_{11}^1 q_1^2 + \alpha_{12}^1 q_1 q_2 + \beta_{111}^1 q_1^3 + \beta_{112}^1 q_1^2 q_2 = \mathbf{\Phi}_1^T \mathbf{f}(t) \quad (3)$$

The NNMs were computed using the Multi-Harmonic Balance algorithm using 5 harmonics and their comparison is shown in Fig. 3. The curve obtained using only the dominant terms matches well with the true NNM over a large range of the motion amplitudes (up to two times the beam thickness). In contrast, if the dominant terms are excluded – then the normal mode is a straight vertical line. This indicates that the structure behaves linearly when these dominant terms are removed from the EOM.

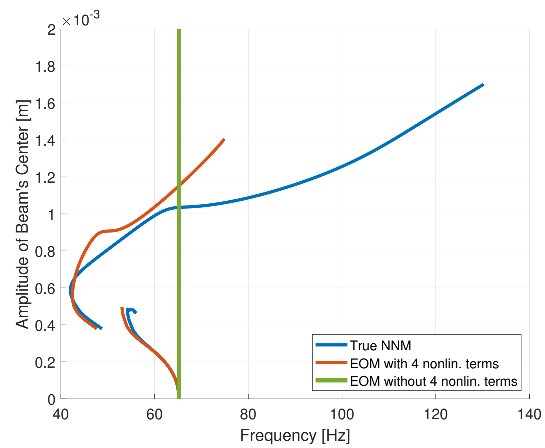


**Fig. 3:** A comparison of the true NNM curve to the one computed using four nonlinear terms pointed out by the NIXO algorithms.

### Conclusion and Future Work

This work briefly discussed a capability of the NIXO algorithms, which could allow them to become an effective black-box nonlinear identification tool. The results presented here prove that the methods can successfully determine the smallest set of terms which should be kept in the nonlinear equation of motion. Since the dynamics of the structure described with the reduces and full sets of terms are comparable - the terms pointed out by NIXO can be considered as dominant in the mechanical system.

In future work, the method will be employed experimentally to identify the dominant nonlinear terms in an equation of motion of flat and curved beams. Then the NNMs of the identified models will be computed and validated against those collected using well-established phase resonance tests.

## References


[1] Kwarta, M., and Allen, M. S., 2020. "Extensions to NIFO and CRP to Estimate Frequency-Independent Nonlinear Parameters". In Proceedings of the 38th International Modal Analysis Conference (IMAC), Houston, TX, USA.