Nonlinear Normal Mode Estimation with Near-Resonant Steady State Inputs

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Abstract

Nonlinear normal modes (NNMs) have been widely used for understanding and characterizing the motion of nonlinear structures, yet current methods to measure them experimentally are time-consuming and not always reliable. Since the structural nonlinearities usually occur when the sample oscillates at high amplitudes, specimens can be damaged or at least develop fatigue cracks when the testing is lengthy. Moreover, the interaction between the shaker and the structure can lead to distortions of the excitation force and can impact the quality of the measured test data. In our previous work, we proposed an NNM estimation algorithm that can help to overcome the issues mentioned above. The approach uses near-resonant data together with an algorithm based on the Single Nonlinear Resonant Mode (SNRM) method to then estimate the NNM backbone. The SNRM algorithm, in its original form, requires vibration modes to be well-separated and assumes no internal resonances between them. This work proposes a possible modification to the algorithm that will allow the modal coupling to be detected as well. The final version of the algorithm will be first tested with data generated numerically using a reduced model of a curved beam experiencing modal interactions. Then the method will be used to estimate the NNMs of a curved steel beam that exhibits significant modal interactions. The results will be validated against those obtained using well-established testing approaches.

Keywords: Nonlinear System Identification, Single Nonlinear Resonant Mode Method, Modal Coupling, Nonlinear Normal Modes, Nonlinear Modal Analysis

Overview of the Basic SNRM Algorithm

The authors' prior work, presented in [1,2], used the Single Nonlinear Resonant Mode method to predict the Nonlinear Normal Mode backbone of a mechanical system experiencing very limited modal coupling. That algorithm is based on the SNRM equation (1), which was first proposed in [3].

$$V^{\text{meas}} \approx V^{model} = \left.\frac{\mathbf{\Phi}_j \mathbf{\Phi}_j^T \mathbf{F}\Omega}{\tilde{\omega}_{0,j}^2 - \Omega^2 + 2i\tilde{\zeta}_j\tilde{\omega}_{0,j}\Omega}\right|_{\substack{\text{pt. of max.}\\ \text{deflection}}} + \sum_{\substack{k=1\\ k\neq j}}^{N_{lin}} \left.\frac{\mathbf{\Phi}_k \mathbf{\Phi}_k^T \mathbf{F}\Omega}{\omega_{0,k}^2 - \Omega^2 + 2i\zeta_k\omega_{0,k}\Omega}\right|_{\substack{\text{pt. of max.}\\ \text{deflection}}}, \tag{1}$$

where:

- V is the complex amplitude of the velocity signal,
- Ω is the forcing frequency,
- $\mathbf{\Phi}_i$, $\omega_{0,i}$, ζ_i are the mode shape, natural frequency and modal damping ratio of the i-th mode, respectively,
- $\mathbf{F}$ is a vector giving the spatial distribution of the sinusoidal excitation force,
- j is the index of the dominant mode,
- N_{lin} denotes the number of relevant linear modes, and
- the quantities marked ($\sim$) vary with the vibration level.

To identify a mechanical system experiencing modal coupling, Eq. (1) has to be modified. A discussion on how this might be done is presented in the next sections. The concepts proposed herein are motivated by the measurements collected in several numerical tests.

Nonlinear Resonant Steady State Response Analysis

The modifications proposed here are motivated by the results collected in a numerical simulation of a single input Force Appropriation test. This test was performed on a simulation model of a *curved beam* with clamped-clamped boundary conditions. The beam was created using 400 shell elements resulting in a total of 3030 DOFs and was reduced to a 2-mode ICE-ROM including modes 1 and 2.

The backbone curve of the first Nonlinear Normal Mode of the beam is shown in Fig. 1. It consists of three segments that were computed separately because the response of the beam is unstable in the vicinity of pairs of points (1G, 2A) and (2E, 3A), which are marked in the figure. The authors suspect that the structure experiences internal resonance near pair (1G, 2A). The reason for instability near the backbone's point of minimum frequency is at this moment unknown and will be investigated.

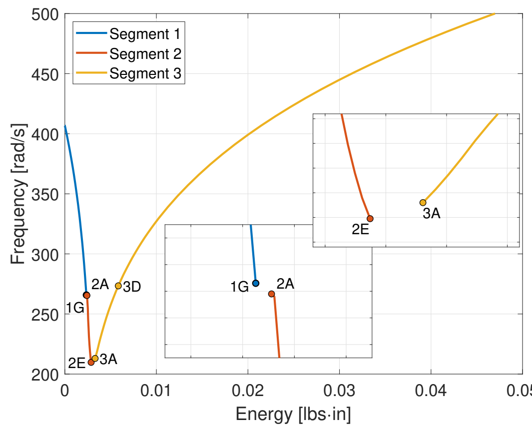

Fig. 1: Segments of the NNM backbone curve presented on frequency-energy plot.

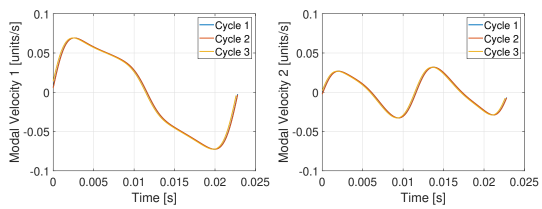

Fig. 2: Modal velocity signals, $\dot{q}_k(t)$, $k \in \{1,2\}$, of the mechanical system oscillating at point 3D (marked in Fig. 1) during three consecutive cycles ($T = \frac{2\pi}{\Omega}$). (The cycles almost overlay.)

The response of the nonlinear part of the mechanical system at point 3D (also marked in Fig. 1), decomposed into modal velocities $\dot{\mathbf{q}}(t)$, is presented in Fig. 2 and Tab. 1. Figure 2 shows time responses of modal velocities, while in Tab. 1 the magnitudes of their Fourier coefficients are presented and compared with one another indicating which modes/harmonics participate the most in the system's response. The quantities $\dot{\mathbf{q}}(t)$ and $\mathbf{v}_j^{nl}(t)$ are defined in Eqs. (2) and (3), respectively.

$$\dot{\mathbf{q}}(t) = \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \vdots \end{bmatrix} = \mathbf{\Phi}^{-1}\mathbf{v}_j^{nl}(t) \tag{2}$$

Table 1 shows that the modal coupling at point 3D takes place between five modes/harmonics. Namely, between mode one occurring as first, second and third harmonics and mode two occurring as second and fourth harmonics. These modes/harmonics are also dominant in the steady-state response of the structure oscillating in the vicinity of point 3D. Thus it might be possible to estimate the NNM parameters based on the near-resonant measurements, even if the structure experiences modal coupling. The next section presents an overview of a concept that could be used to modify the original SNRM formulation so that it can successfully identify a nonlinear mechanical system using near-resonant response data such as that shown here.

Tab. 1: Fourier coefficients magnitudes of the first five harmonics of modal velocities presented in Fig. 2 and their ratio to the maximal coefficient value (expressed in %). Values marked with blue correspond to the modes/harmonics which are considered to exhibit modal coupling.

	Ω		2Ω		3Ω		4Ω		5Ω	
$\dot{q}_1(t)$	**6.77e-02**	**(100.00)**	**1.59e-02**	**(23.45)**	**1.37e-02**	**(20.29)**	3.52e-03	(5.19)	2.58e-03	(3.81)
$\dot{q}_2(t)$	1.04e-03	(1.54)	**2.84e-02**	**(41.89)**	1.87e-03	(2.76)	**6.48e-03**	**(9.57)**	9.61e-04	(1.42)

Discussion on the SNRM Algorithm Extension

A generalized form of the SNRM model function is presented in Eq. (3). Rather than expressing the response using a single complex amplitude, as in Eq. (1), it considers the motion over a certain time. Hence, one could include the sub- or higher-harmonics in the system's response (as e.g. indicated in Tab. 1). This form also brings the model closer to the original Nonlinear Normal Modes definition, which introduces them as (non-necessarily synchronous) periodic motions of the conservative system [4].

$$\mathbf{v}^{meas}(t) = \mathbf{v}_j^{nl}(t) + \underbrace{\mathrm{Re}\left\{\sum_{\substack{k=1 \\ k\neq j}}^{N_{lin}} \frac{\mathbf{\Phi}_k\mathbf{\Phi}_k^T\mathbf{F}\Omega\ e^{i\Omega t}}{\omega_{0,k}^2 - \Omega^2 + 2i\zeta_k\omega_{0,k}\Omega}\right\}}_{\mathbf{v}_j^{lin}(t)}, \tag{3}$$

where:

- $\mathbf{v}^{meas}(t)$ is the full-field velocity response of the structure (measured experimentally or numerically),
- $\mathbf{v}_j^{nl}(t)$ is the full-field nonlinear velocity response of the structure oscillating near the j-th NNM and
- $\mathbf{v}_j^{lin}(t)$ is a term responsible for modeling the response of the system far from the j-th NNM.

One of the possible concepts of how to express the quantity $\mathbf{v}_j^{nl}$ is shown in Eq. (4). This formula allows for modeling the response with several modes and/or harmonics. Additionally, it has certain similarities to the nonlinear term from Eq. (1), which facilitates a physical interpretation of the new quantities introduced in (4).

$$\mathbf{v}_j^{nl}(t) = \mathrm{Re}\left\{ \sum_{n=1}^{N_{nl}} \frac{\mathbf{\Phi}_j(n)\mathbf{\Phi}_j^T(n)\ \mathbf{F}\ \Omega\ e^{i[h_j(n)]\Omega t}}{\tilde{\omega}_{0,j}^2 - \Omega^2 + 2i\ \tilde{\zeta}_j(n)\ \tilde{\omega}_{0,j}\Omega} \right\} \tag{4}$$

The new quantities introduced in Eq. (4) are defined as follows:

- N_{nl} indicates how many coupling terms should be considered in the system identification process,
- $\tilde{\zeta}_j(n)$ is the damping ratio corresponding to mode $\mathbf{\Phi}_j(n)$ which is expected to occur in the system response. If $\tilde{\zeta}_j(n)$ is a large number than the mode $\mathbf{\Phi}_j(n)$ is negligible in the system's response.
- $h_j(n)$ indicates if the mode $\mathbf{\Phi}_j(n)$ vibrates with the forcing frequency Ω ($h_j(n) = 1$), or if it appears as a sub- or higher harmonic ($h_j(n) \neq 1$).

In the case study presented in the previous section, which focuses on the motion near the first NNM ($j = 1$), the quantities introduced above should be given the following values: $N_{nl} = 5$, $\mathbf{\Phi}_1(n) = \left[\mathbf{\Phi}_1\ \mathbf{\Phi}_1\ \mathbf{\Phi}_1\ \mathbf{\Phi}_2\ \mathbf{\Phi}_2\right]$, $h_1(n) = \left[1\ 2\ 3\ 2\ 4\right]$ and $\tilde{\zeta}_1(n) = \left[\tilde{\zeta}_{1,1}\ \tilde{\zeta}_{1,2}\ \tilde{\zeta}_{1,3}\ \tilde{\zeta}_{1,4}\ \tilde{\zeta}_{1,5}\right]$. The damping ratios of the coupled modes ($\tilde{\zeta}_{1,k},\ k \in \{1,\dots,5\}$) could be modeled as unknown functions of vibration level, with values known when the system vibrates at low amplitudes. At low vibration levels, the response of the structure is dominated by the underlying linear system. Thus, $\tilde{\zeta}_{1,1} = \tilde{\zeta}_1^{lin}$ and $\tilde{\zeta}_{1,k},\ k \in \{2,\dots,5\}$ should be given large enough values, so that the contribution of the their pseudo-modes to the system's response can be treated as negligible.

The extension to the SNRM algorithm discussed briefly in this section is one of several possible concepts the authors are currently investigating. The final version of the model function (4) and the discussion on its correctness from the physical standpoint as well as the ability to capture modal interactions experienced by the oscillating structure will be presented at the conference.

Conclusion and Future Work

This work briefly discussed one possible modification to the SNRM algorithm that would enable it to capture modal coupling. The main goal of this extension is to estimate the Nonlinear Normal Mode backbone curve and additionally detect the modal interaction. The authors are currently investigating variations on the model function in (4) in order to determine which to implement in the final version of the system identification algorithm.

In future work, the method will be tested numerically using a ROM of a curved beam, which experiences significant modal coupling. Then the algorithm will be used to identify the NNMs of a curved steel beam that exhibits modal interactions. The results will be validated against those obtained using well-established testing approaches.

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