Geometrical Theory of Nonlinear Modal Analysis

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Overview

- Motivation and backgrounds

- Nonlinear modes of vibrations
  - Three definitions
  - Instantaneous Center Manifold (ICM)

- Calculation of nonlinear modes
  - Solving for ICM analytically
  - Averaging and collocation methods
  - Averaging $\oplus$ collocation (MMC)

- Stability and bifurcation of nonlinear modes
  - Floquet theory
  - Validation of stability analysis

- Connecting functions
  - Definition of connecting functions
  - Bi-directionally linear connecting functions
  - Calculation of connecting functions
  - Linear approximation of connecting functions

- Conclusions

Motivations and Backgrounds

- History

  - "In his work on dynamics, Poincare was led to focus attention primarily upon the periodic motions. He conjectured that any motion of a dynamical system might be approximated by means of those of periodic type, i.e. that the periodic motions to be densely distributed among all possible motions; and it became a task of the first order of importance for him to determine what the actual distribution of the periodic motions was, so as to prove or disprove his conjecture." [1]

  - This conjecture was proved for linear systems by Hilbert (known as spectral theory) and is the foundation of every technique/method used in modal analysis.

- Primary objectives

  - The primary objective of this work is to provide new insights on how to calculate all periodic solutions of a class of nonlinear systems efficiently and then use them to arbitrarily accurately approximate any solution of such systems.

[1] Birkhoff, George D, "On the periodic motions of dynamical systems", *Acta Mathematica* 50, 1 (1927), pp. 359--379.

Future Applications

- Predicting the life cycle and guiding design changes
  - Engineers prefer to design systems to be linear, many systems are just intrinsically nonlinear or the linear designs may be suboptimal with respect to the intended purpose.

  - By altering the design the life can change by orders of magnitude.

- Predicting the behavior of nonlinear dynamical systems

  - Accurate calculation of periodic solutions and their bifurcations are required for determining the path of (long-period) comets.
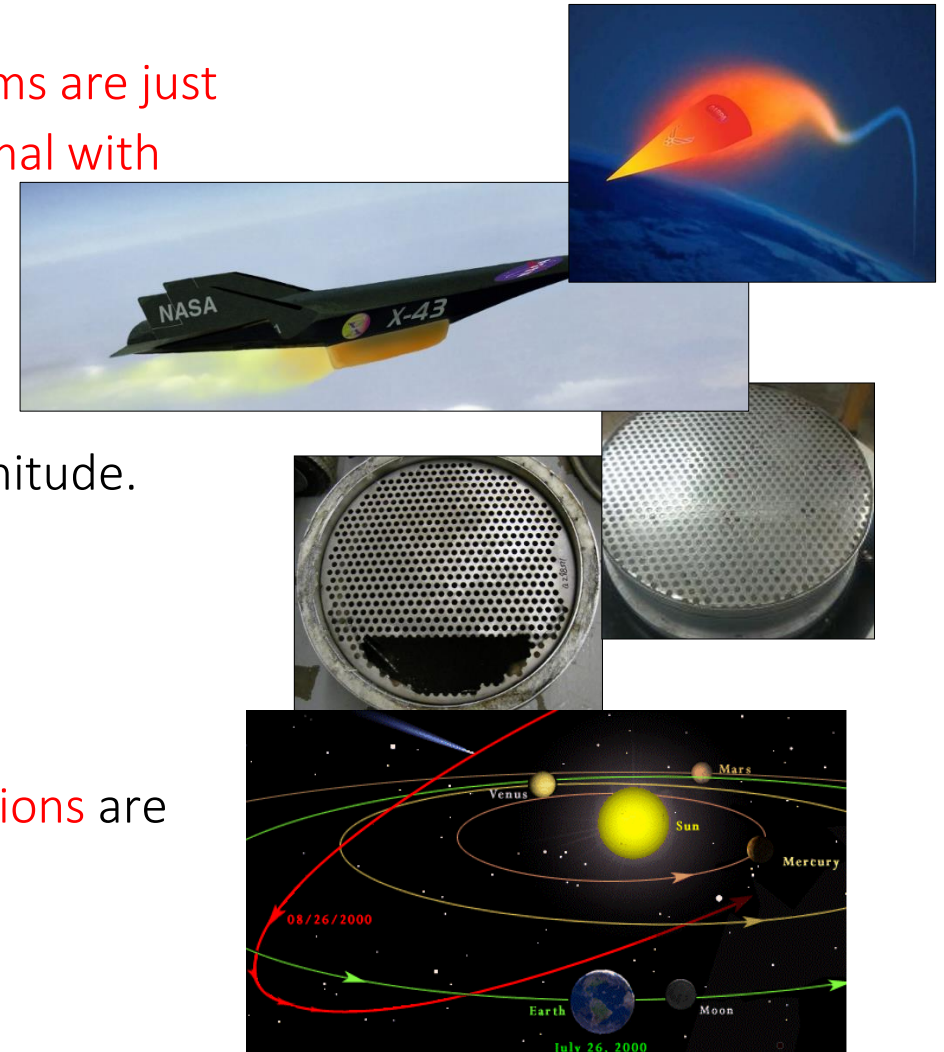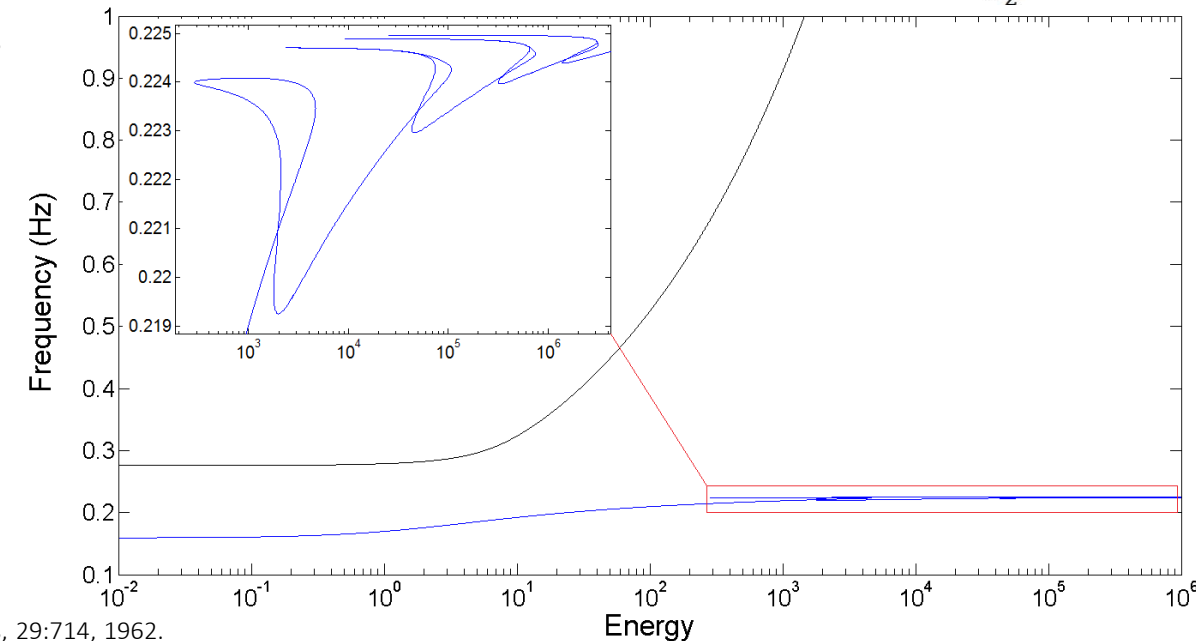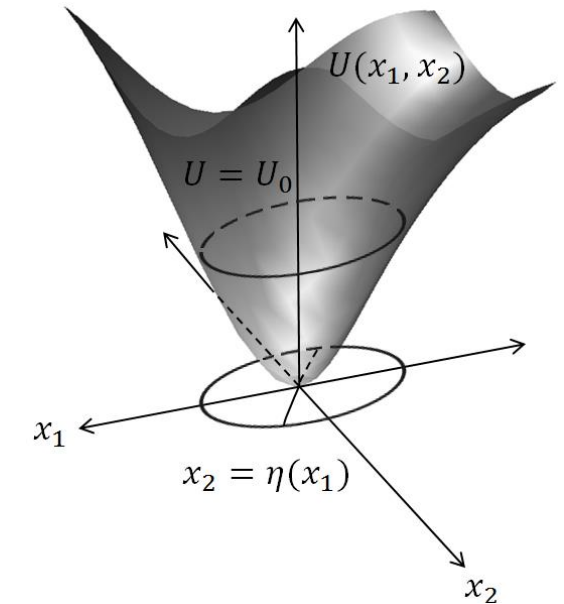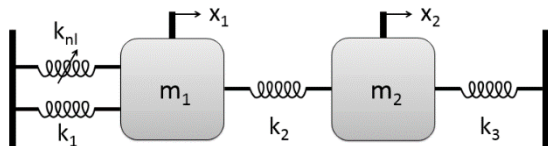
Nonlinear modes are periodic solutions.

- Rosenberg defined a nonlinear mode as a one-dimensional functional relation between coordinates of a periodic solution $\boldsymbol{\eta}(x_1)$, i.e.

  - Any solution: $\boldsymbol{\eta}(x_1(t)) = \boldsymbol{\eta}(x_1(t + T))$
  - Synchronous: $\boldsymbol{\eta}(x_1(0)) = 0$
  - Orthogonal to equipotential curves [2].

- Vakakis modified Rosenberg's definition to any periodic solution $\boldsymbol{x}(t)$ i.e.

  - Any solution: $\boldsymbol{x}(\text{t}) = \boldsymbol{x}(t + T)$ [3].

[2] R.M. Rosenberg. On normal vibrations of a general class of nonlinear dual-mode systems. Journal of Applied Mechanics, 29:714, 1962.
[3] A. F. Vakakis. Analysis and identification of linear and nonlinear normal modes in vibrating systems. PhD thesis, California Institute of Technology, 1990.

Nonlinear modes are two-dimensional functional relations.

- Shaw and Pierre defined a nonlinear mode as a two-dimensional time independent functional relation that satisfies the governing equations of the system i.e. $\mathbf{\Gamma}(x_1, \dot{x}_1)$ that

  - is invariant (time independent), $\Gamma_i = a_{i1}x_1 + a_{i2}\dot{x}_1 + a_{i3}x_1^2\dot{x}_1 + a_{i4}x_1\dot{x}_1^2 + \cdots$

  - satisfies the governing equations of motion, i.e. $\dfrac{d^2\mathbf{\Gamma}}{dt^2} = \boldsymbol{f}\left(\mathbf{\Gamma}, \dfrac{d\mathbf{\Gamma}}{dt}\right)$ [5].
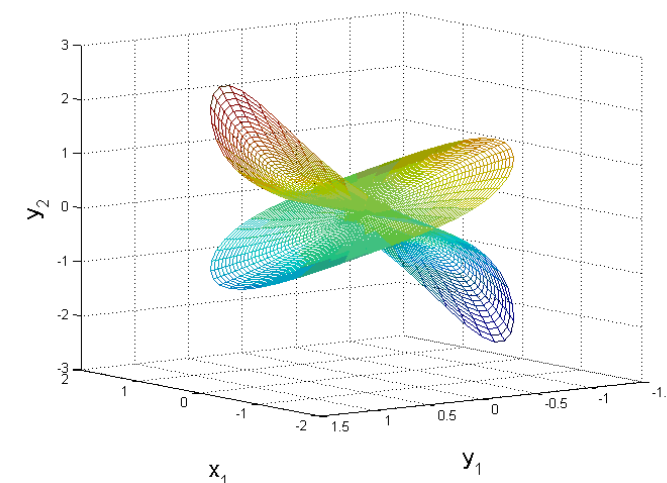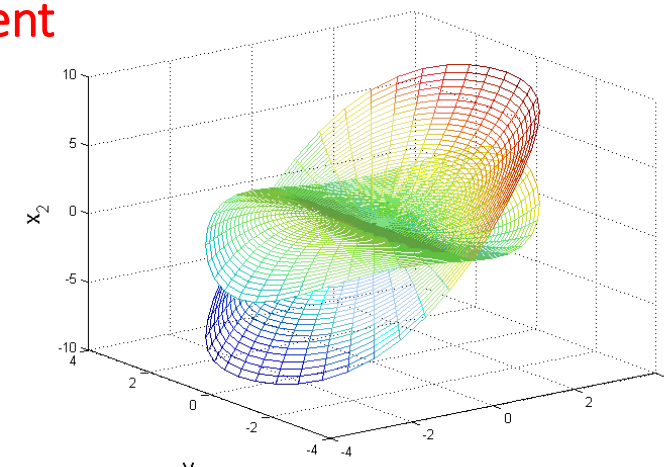
    - They are tangent to the vector field at its fixed point.

    - When $\mathbf{\Gamma}$ can be a manifold?
    - Is $\mathbf{\Gamma}$ invariant? Why is $\mathbf{\Gamma}(x_1, \dot{x}_1)$ tangent to the vector field?
    - Why only fixed points?

[4] S.W. Shaw and C. Pierre. Non-linear normal modes and invariant manifolds. Journal of Sound and Vibration, 150(1):170173, 1991.

This work presents a new definition for invariance leads to a unified definition for invariant manifolds of both fixed points and periodic solutions.

- Γ is an invariant manifold under f if and only if f is always in the tangent bundle of Γ.

- We proved that a manifold $\Gamma = \begin{bmatrix} U_i \\ \dfrac{dU_i}{dt} \end{bmatrix}$ is invariant under the system $\begin{array}{l} \dot{x} = y \\ \dot{y} = f(x,y) \end{array}$, if and only if
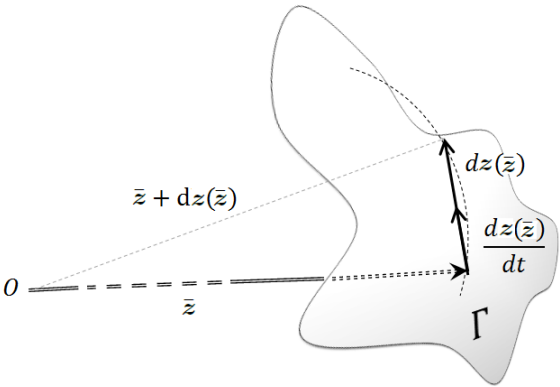
$$D_{\phi\phi}^{\omega\omega}U_i + 2D_{a\phi}^{\rho\omega}U_i + D_{aa}^{\omega\omega}U_i + D_{\phi}^{\dot\omega}U_i + D_a^{\dot\rho}U_i = g_i$$

$$w = H_x \qquad a_i(t) = (x_i^2 + w_i^2)^{\frac{1}{2}} \qquad g_i(a,\phi) = f_i(x(a,\phi), y(a,\phi))$$

$$z = H_y \qquad \phi_i(t) = \arg(w_i, x_i)$$

- Γ's are especially interesting when calculated around equilibrium, i.e. fixed point and periodic solutions, of f .
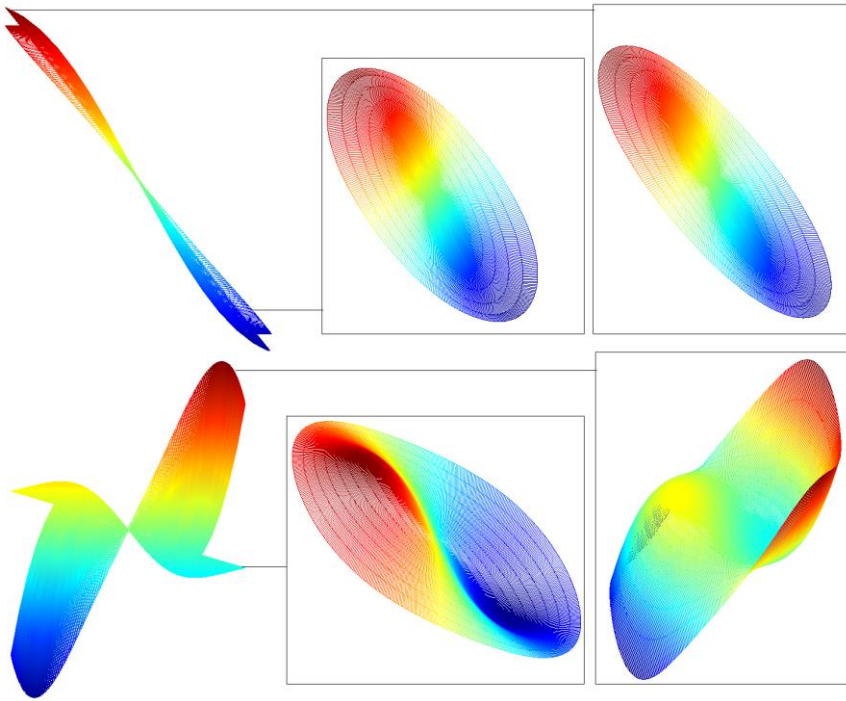
Local invariant manifolds of a nonlinear system can be obtained without an explicit localization of the system.

- Therefore all invariant manifolds $\mathbf{\Gamma}$ of f can be obtained by solving the same set of PDE's, weather they are defined around a fixed point

- or an (unknown) periodic solution

Additional questions that are not answered her: Feel free to ask me:

- Why is called ICM?
- What is the relationship between the center manifold of a system (if it exists) and its ICMs?
- When do ICMs become globally inseparable manifolds?
- Do we need higher dimensional ICMs? Do they exist?
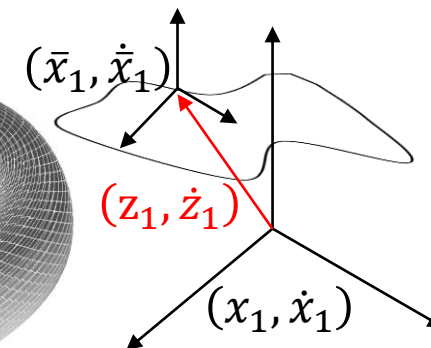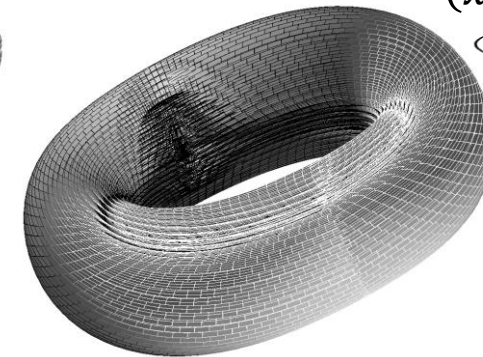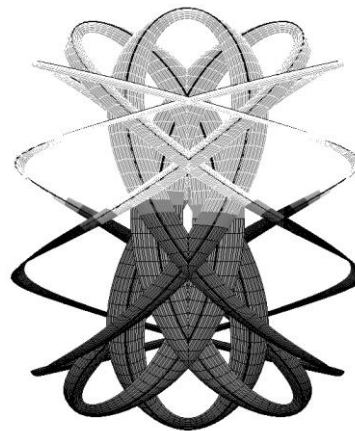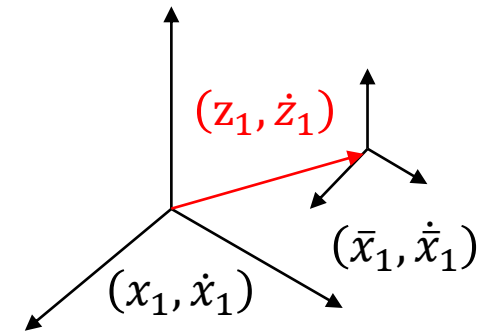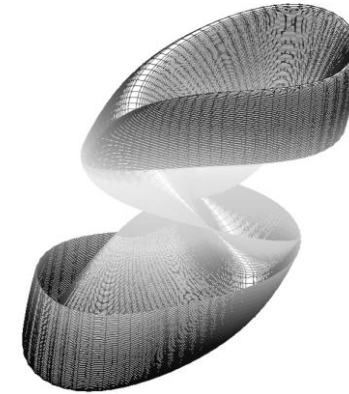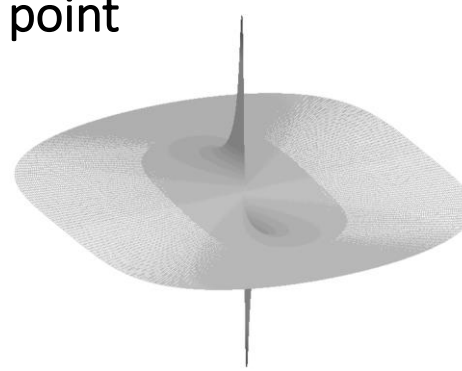
An analytical method was presented to solve the governing PDE's of each ICM.

- A combination of an averaging method (harmonic balance) and a nonlinear (algebraic) elimination technique was used.

- This way by identifying (only) the independent coordinates, i.e.

$$U_i = \sum_{m \in \mathbb{N}} \left[A_m^{(i)} \cos(m\phi) + B_m^{(i)} \sin(m\phi) \right]$$

one also identifies the invariant manifold (functional relation).
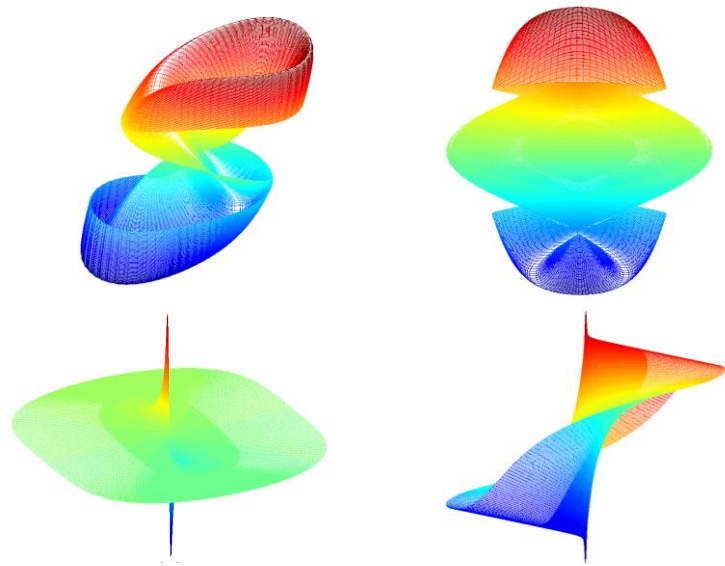
This method is not scalable!
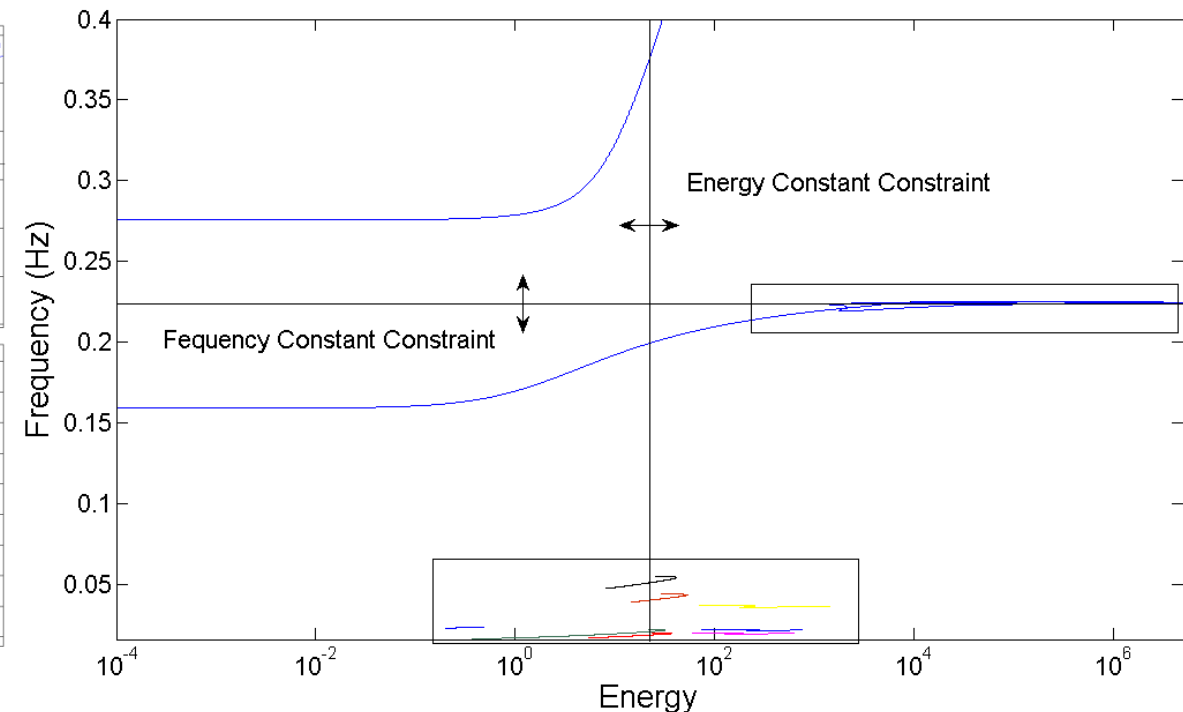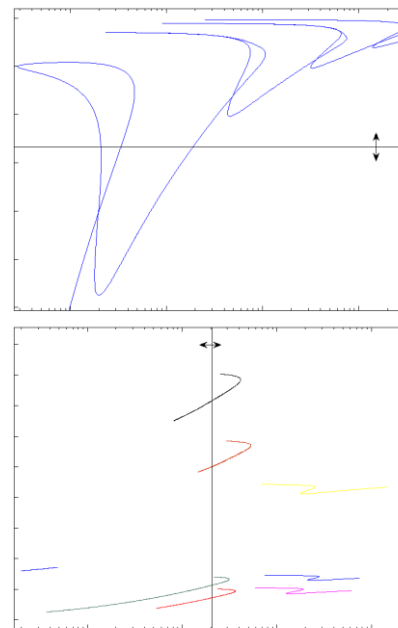
Two classes of methods currently exist. Can a new method, that combines the befits of both averaging and collocation methods without any of their drawbacks, be developed?

- **Averaging methods** try to make a parametric periodic function satisfy the governing equations of the system.

  - One example: Harmonic balance.

  - We have to integrate the system analytically!

  - They are not scalable.

- **Collocation methods** integrate the system numerically to check the periodicity of the solution.

  - They are scalable: we can integrate numerically.

  - They are computationally expensive: we often have to Integrate the system over and over.

  - They are sensitive to the initial conditions.
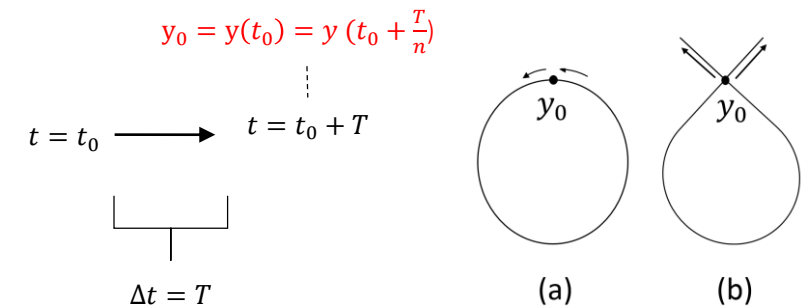
$$y_0 = y(t_0) = y\left(t_0 + \frac{T}{n}\right)$$

$$t = t_0 \longrightarrow t = t_0 + T$$

$$\Delta t = T$$

(a)   (b)

Averaging methods try to make a parametric periodic function satisfy the governing equations of the system!

- Assume a sum of (orthogonal) periodic functions

$$s(t) = \sum_{k=1}^{n} \alpha_k p_k(t)$$

- Approximates a solution of the system, i.e.

$$\ddot{s}(t) - f\big(s(t)\big) = \mathbf{0}$$

- To find the unknown coefficients

$$\int_0^T \ddot{s}.p_k(\tau)\, d\tau = \int_0^T f(s).p_k(\tau)\, d\tau$$

- Results in an algebraic system of equations

$$g(\alpha_1, \dots, \alpha_1) = \mathbf{0}$$

- The right-hand-side is a weighted average of the system

$$\int_0^T f(s).p_k(\tau)\, d\tau = \frac{T \sum_{m=0}^{M}\left[f\big(s(\frac{mT}{K})\big) p_k(\frac{mT}{K})\right]}{M}, M \to \infty$$

In effect, averaging methods minimize a weighted average of the difference in acceleration over infinitely many points.

- Integration is just a weighed averaging

$$\int_0^T \boldsymbol{f}(\boldsymbol{s}).\,\boldsymbol{p}_k(\tau)\,d\tau = \frac{T\sum_{m=0}^{M}\left[f\left(\boldsymbol{s}\left(\frac{mT}{M}\right)\right)\boldsymbol{p}_k(\frac{mT}{M})\right]}{M}, M \to \infty$$

$$\int_0^T \ddot{\boldsymbol{s}}.\,\boldsymbol{p}_k(\tau)\,d\tau = \frac{T\sum_{m=0}^{M}\left[\ddot{\boldsymbol{s}}\left(\frac{mT}{M}\right)\boldsymbol{p}_k(\frac{mT}{M})\right]}{M}, M \to \infty$$

- Averaging methods just minimize a weighted average of the difference in accelerations, i.e.

$$\sum_i \frac{T}{M}\left[\ddot{\boldsymbol{s}}(t_i) - \boldsymbol{f}\big(\boldsymbol{s}(t_i)\big)\right]\boldsymbol{p}_k(t_i) = \boldsymbol{0}\quad t_i = \frac{iT}{M}, i = 0,\dots M \to \infty$$

- The periodic sum approximates a solution of the system because, in average, it matches with it at infinitely many points in one period

    - Indefinite answer

    - We have to do it analytically!

Collocation methods integrate the system numerically to check the periodicity of the solution.

- The solution of the system is periodic because it crosses itself after ONE period.

- Definition of periodic functions + Uniqueness of solutions

$$y(t) = y_0 + \int_0^t g(y(\tau))d\tau, \qquad y = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, g = \begin{bmatrix} \dot{x} \\ f(x) \end{bmatrix}$$

- Find $\{T, y_0\}$ that $y(T) = y_0$

  - We can integrate numerically

  - One constraint → definite answer

  - Integrate the system over and over

$$y_0 = y(t_0) = y\left(t_0 + \frac{T}{n}\right)$$

$t = t_0 \longrightarrow t = t_0 + T$

$\Delta t = T$

y_0

(a)

y_0

(b)
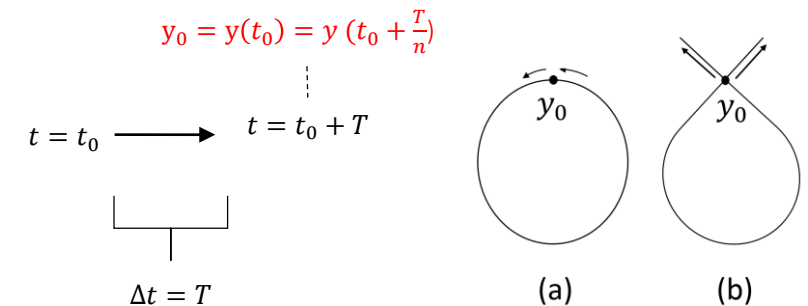
Averaging Collocation: Can we replace the infinitely many integration points with only a few collocation points?

Averaging:

    One time Analytical integration

    Indefinite answer

Collocation:

    Many times Numerical integration

    Definite answer

Recall our conditions

$$\sum_i \frac{T}{M}\left[\ddot{s}(t_i) - f(s(t_i))\right]p_k(t_i) = 0 \quad t_i = \frac{iT}{M}, i = 0, \ldots M \to \infty$$

$$y(T) - y_0 = \int_0^T g(y(\tau))d\tau = 0, \qquad y(0) = y_0$$

- Can we replace the average sum with a set of much stronger collocation conditions?

- Can we do it for only a few m pairs of collocation points $m \ll M$, i.e. without any Integration?

Multi-harmonic Multiple-point Collocation (MMC) provides the solution!

- MMC uses a Fourier series as the parametric periodic sum

$$s(t) = C + \sum_{k=1}^{n} A_k cos(\frac{2\pi kt}{T}) + B_k \sin(\frac{2\pi kt}{T})$$

to minimize the collocation condition

$$\min_{\{A,B,C,T\}} \sum_m \left\{ \left\| \underbrace{\left[f\left(s\left(\frac{mT}{K}\right)\right) - \ddot{s}\left(\frac{mT}{K}\right)\right]}_{d_m(\frac{mT}{K})} \right\| + \left\| \underbrace{f\left(s\left(\frac{mT}{K}+T\right)\right) - \ddot{s}\left(\frac{mT}{K}+T\right)}_{d_m(\frac{mT}{K}+T)} \right\| \right\} = 0$$

- **I proved that MMC converges iff**

   f **is monotonic in the neighborhood of collocation points.**

- Comes in two flavors:

  - Newtonian: Very fast but with a very small radius of convergence

  - Steepest descent (with Wolfe condition) : Slower but has a very large radius of convergence
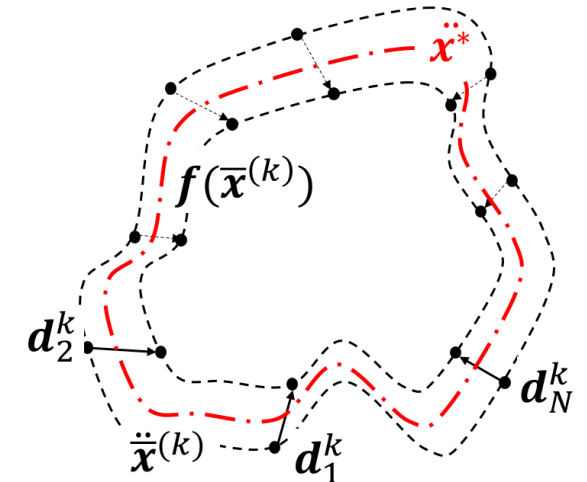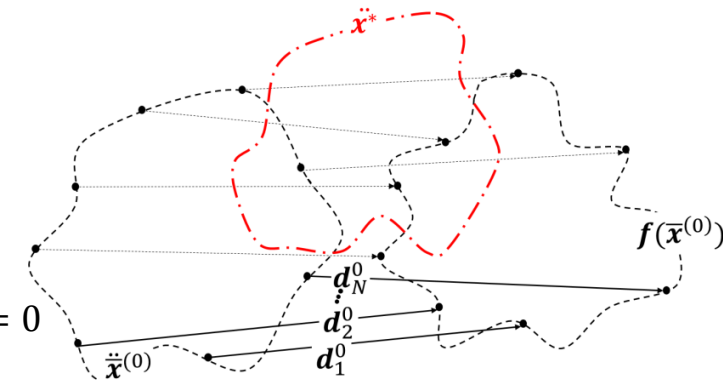
Homogenous convergence refers to the cases where all collocation points converge to the same solution.

- All points converge on the same solution.

- It has a very large radius of convergence (in the steepest descent flavor)!
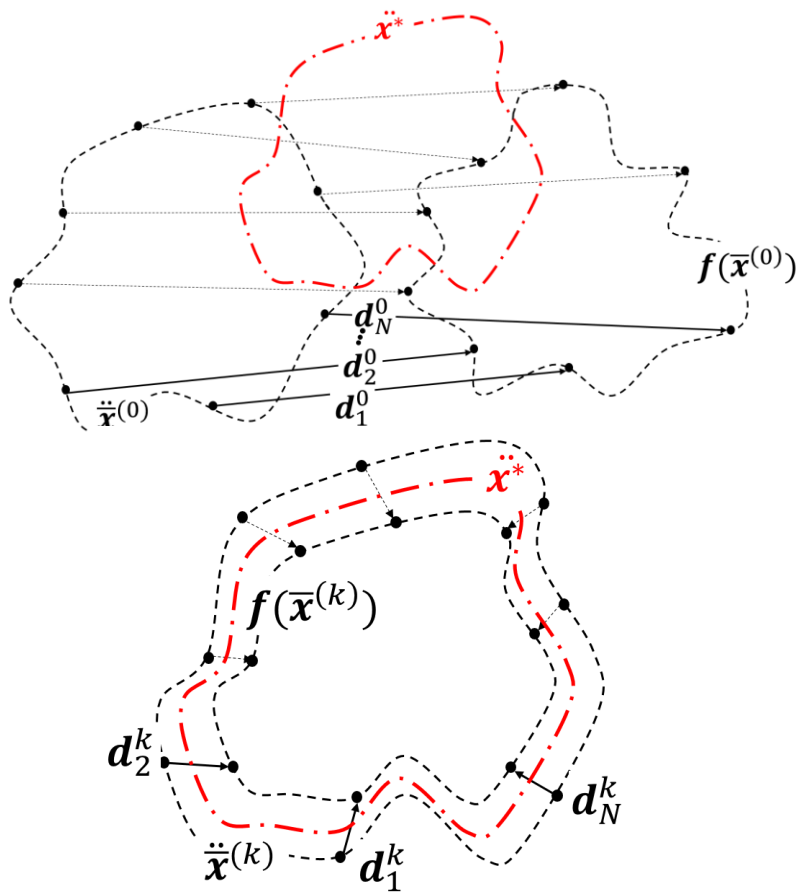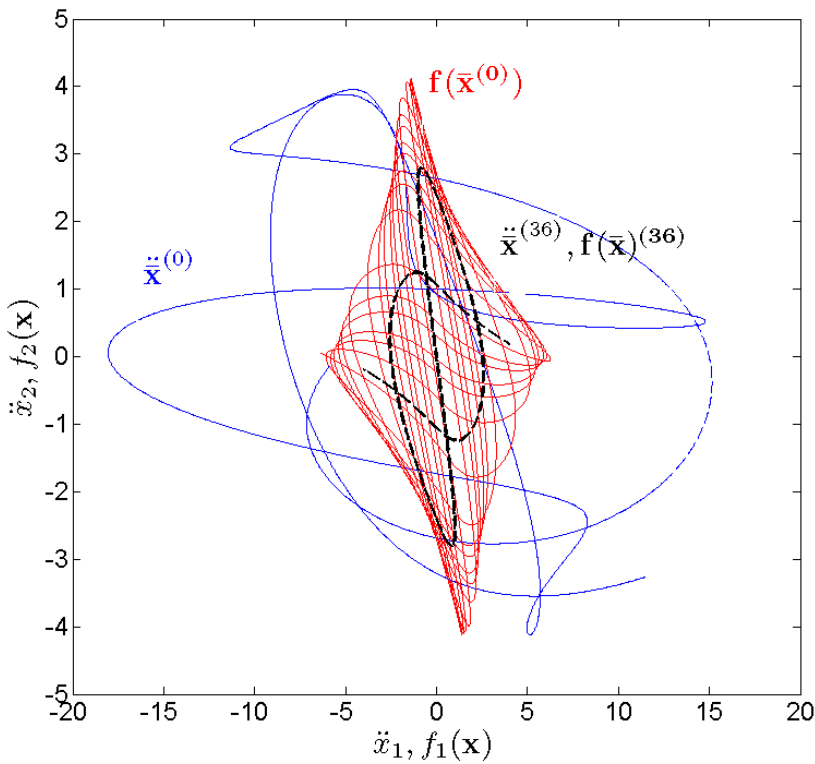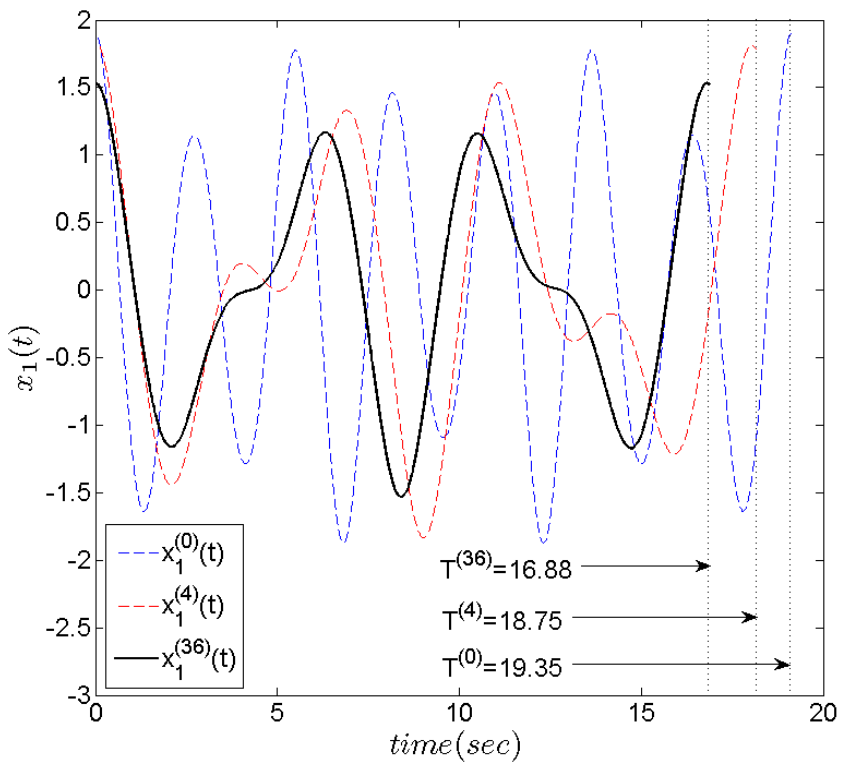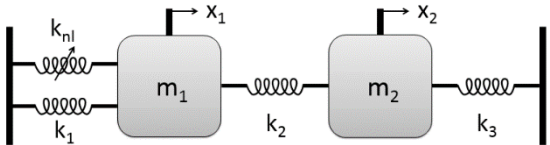
An added benefit: a heterogeneous convergence is possible!

- Without integration, there is no constraint to force the points to be on one solution, i.e. collocation points can converge on different solutions.
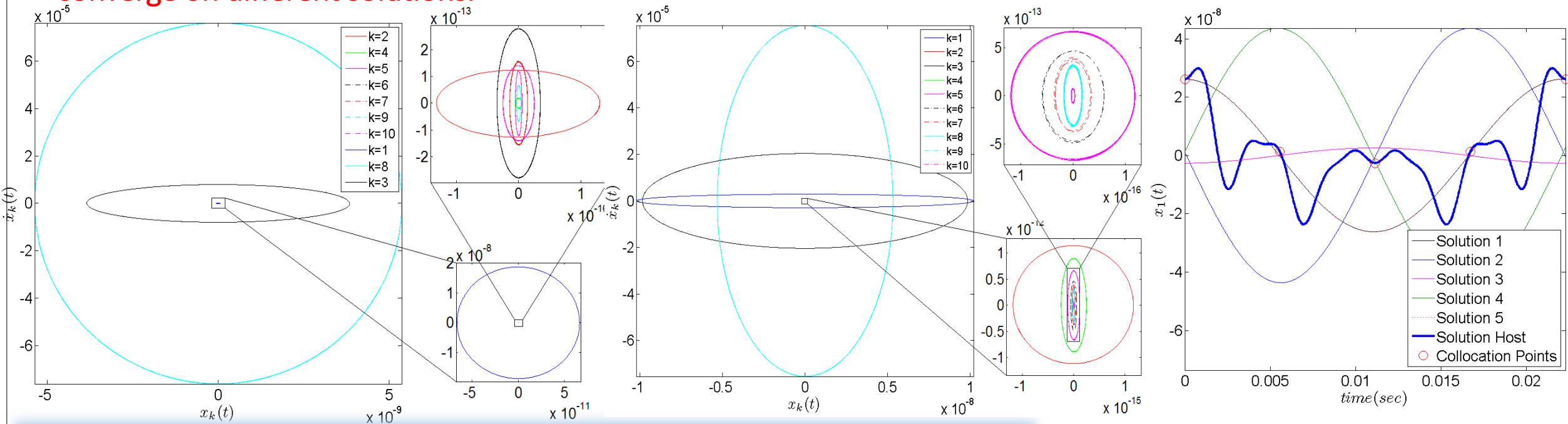
Additional questions:

- What if f is not monotonic around a collocation point?

- Is MMC the only combination of averaging and collocation method possible?

- Can one use continuation with MMC?
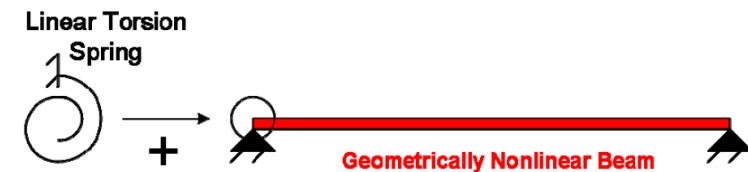
Which solutions are stable? When will we jump from one to another? The main tool for stability analysis of periodic solutions is Floquet theory.

- Stability analysis is important in

  - Design, i.e. identifying the range of safe operation, of nonlinear systems.

  - In experimental identification of nonlinear systems, to predict or explain which set of nonlinear modes can or cannot be experimentally excited.

  - Bifurcation analysis of periodic solutions of nonlinear system.

- Can be performed using Lyapunov or Poincare or Floquet methods

  - All provide interchangeable results. Floquet methods can be implemented much easier!

- Floquet stability

  - Linearize the system about the periodic solution.

  - Determine the growth of a small perturbation around the periodic solution.
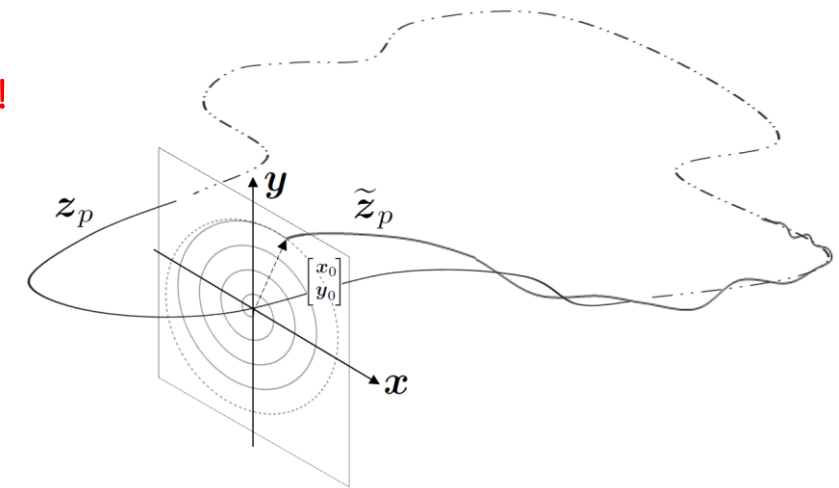
New algorithm by Ardeh-Allen validates near-zero Floquet exponents and returns accurate stability analysis results!
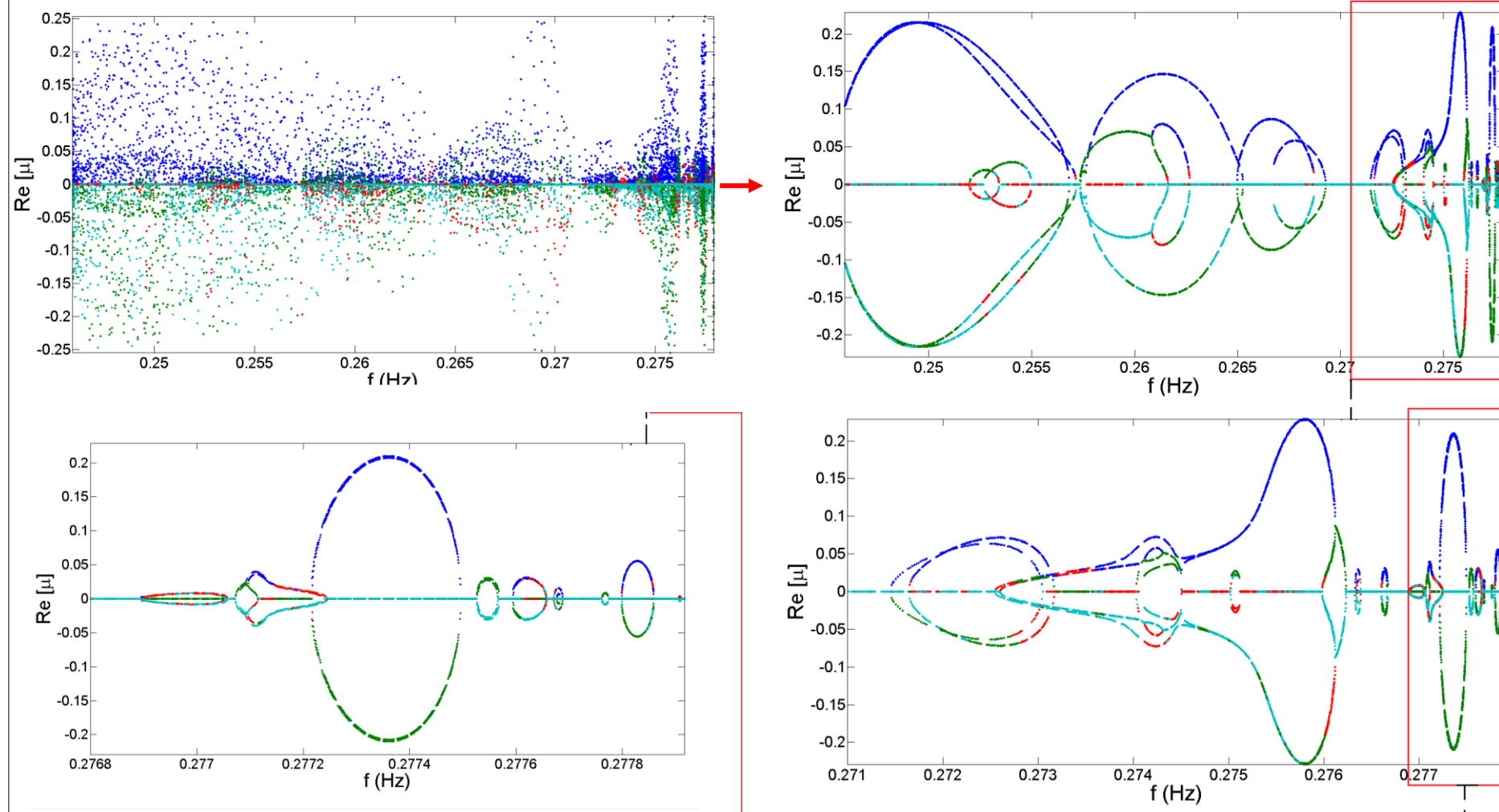
Additional questions:

- Why are near-zero Floquet exponents important?

- How efficient is this algorithm?

- Can this algorithm improve the efficiency of current algorithms of finding periodic solutions?

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- Stability and bifurcation of nonlinear modes
  - Floquet theory
  - Validation of stability analysis

- **Connecting functions**
  - Definition of connecting functions
  - Bi-directionally linear connecting functions
  - Calculation of connecting functions
  - Linear approximation of connecting functions

- Conclusions

Superposition describes a set of two properties.
General solution is a function with a specific domain and codomain!

- Linear Systems: linear System of Autonomous, Homogenous Second Order ODE's

$$\ddot{x} = f(x) = Ax$$

- Additivity

  If q_1 and q_2 are solutions, so is $q_1 + q_2$.

- Homogeneity

  If q_1 is a solution, so is $\alpha q_1, \ \alpha \in \mathbb{R}$.

- Additivity and Homogeneity → Superposition

  If $q_i \ i = 1, \ldots, N \in \mathbb{N}$ are solutions, so is every $F(q_1, \ldots, q_N) = \sum_i \alpha_i q_i$

- The general solution of a linear systems represents any solution of the system as a superposition of a finite number of solutions.

General solution of a linear systems exists because linear systems accept superposition.

- General solution is a function that can represents all solutions of the system as superposition of a finite number of solutions, i.e. a function $\mathbf{F}(\mathbf{q}_1, \dots, \mathbf{q}_N)$ such that

  - F spans the entire set of solutions of the system.

    - For linear systems, if $\mathbf{q}_i$ are linearly independent,

    - $\mathbf{F}(\mathbf{q}_1, \dots, \mathbf{q}_n) = \sum_{i=1}^{n} \boldsymbol{q}_i$ spans the entire set of solutions of the system.

    - $\mathbf{q}_i$ can be linear modes of the system.

  - The set $\{\mathbf{q}_1, \dots, \mathbf{q}_N\}$, must be a finite set.

    - There are infinitely many linear modes.

    - But the system is Homogeneous: we can normalize them! The finite set can be the set of "Linear Normal Modes".

An example: superposition dictates the form of the general solution for a 2DOF linear oscillatory system.

- Consider the 2DOF linear system

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

  - The system has two infinite sets of linear modes.

  - The cross section of these two sets with the plane $t = 0$ generates two lines.

    - The (evolution of) unit vectors along each branch are the linear normal modes.

  - Without the homogeneity property (i.e. normalization), no general solution could exist, because the set of basis vector would be infinite.
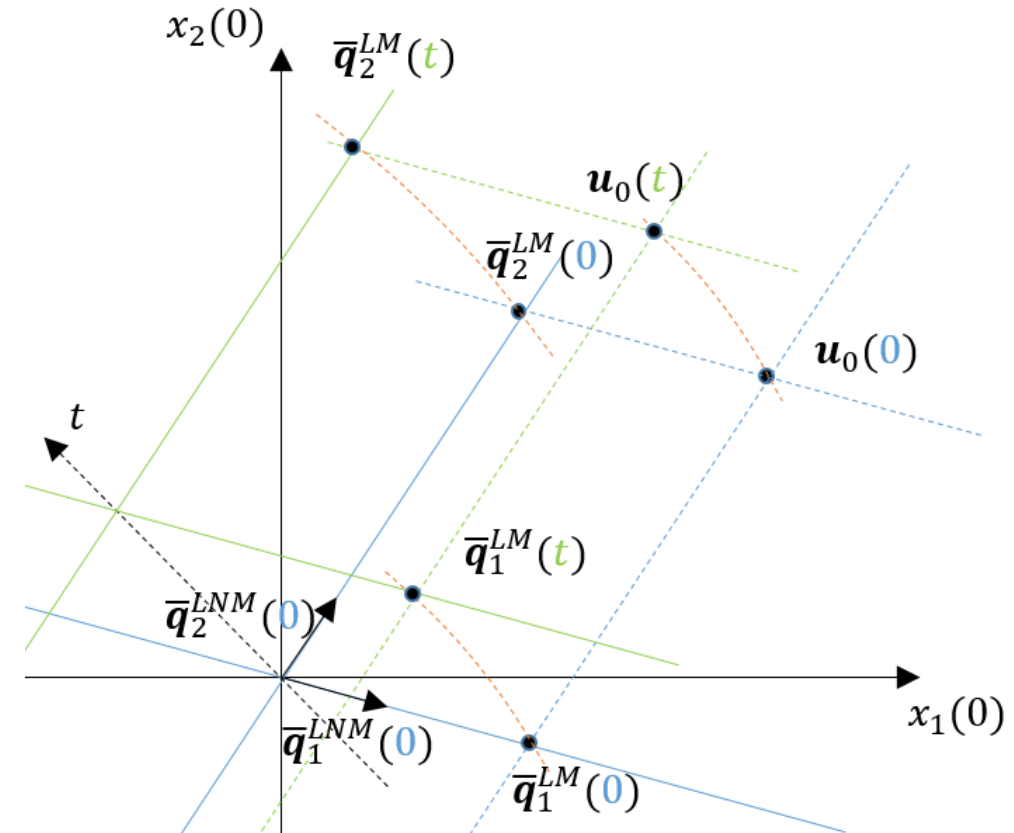
- $\boldsymbol{u}_0(t) = \overline{\boldsymbol{q}}_1^{LM}(t) + \overline{\boldsymbol{q}}_2^{LM}(t)$

- $\boldsymbol{u}_0(0) = \alpha_1 \overline{\boldsymbol{q}}_1^{LNM}(t) + \alpha_2 \overline{\boldsymbol{q}}_2^{LNM}(t)$

- $\alpha_i = \left\| \overline{\boldsymbol{q}}_i^{LM}(0) \right\|$

Connecting functions are like the general solution, i.e. they're functions, but without the constraints on their domain and codomain.

- Connecting Function

  - Any function of solutions of the system that is also a solution of the system.

    Defined as any $F(q_1, \ldots, q_N), N \in \mathbb{N}$ that satisfy $\frac{d^2 F}{dt^2} = f(F)$ are connecting functions.

- Connecting Function Vs. Superposition

  - Superposition defines a family of linear, homogenous connecting functions.

- Connecting Function Vs. General Solution

  - If there exists a family, F, of a finite $\{q_1, \ldots, q_N\}$ that spans the entire set of solutions of the system, then it is a general solution.

Lie proved that no general solution exists in the absence of superposition [5], but it does not mean that a nonlinear connecting function does not exist.

- Without superposition, general solution Does Not Exist (Except for Riccati Equations)!

  - Lie Theorem: For any finite $\{q_1, \dots, q_N\}$ there exist no function, F, that can span the entire set of solutions.

- What About Connecting Functions?

  - The set of nonlinear modes of (eigensolutions of) nonlinear systems, i.e. $\{q_1, \dots, q_N\}$, is infinite!

  - The set of nonlinear modes cannot be normalized (nonlinear systems are not homogenous)!

  - If we relax the finiteness condition and global coverage of F:

    - Is there a nonlinear local connecting function?

[5] S. Lie and G. Scheers. Vorlesungen uuber continuierliche gruppen mit geometrischen und anderen anwendungen, edited and revised by g. Scheers, Teubner, Leipzig, 1893. 5.1.3

Nonlinear connecting functions exist and can be considered as local general solutions!

Additional questions:

- I proved that if a nonlinear local connecting function exists, it must satisfy

  - For a conservative system $\boldsymbol{f}$ with solutions $\{\boldsymbol{q}_1, \dots, \boldsymbol{q}_N\}$

  - $\boldsymbol{\Phi} = \boldsymbol{\Phi}(\boldsymbol{q}_1, \dots, \boldsymbol{q}_N, \dot{\boldsymbol{q}}_1, \dots, \dot{\boldsymbol{q}}_N)$

  - $\mathbb{J}_{\boldsymbol{q}_i}\left(\boldsymbol{\nabla}_{\boldsymbol{q}_j} F_k\right) = \boldsymbol{0}$ and $\mathbb{J}_{\dot{\boldsymbol{q}}_i}\left(\boldsymbol{\nabla}_{\dot{\boldsymbol{q}}_j} F_k\right) = \boldsymbol{0}$

  - $\boldsymbol{\Phi}(\boldsymbol{q}_1, \dots, \boldsymbol{q}_N, \dot{\boldsymbol{q}}_1, \dots, \dot{\boldsymbol{q}}_N) = \boldsymbol{d} + \sum_i A_i\, \boldsymbol{q}_i + \sum_i B_i\, \dot{\boldsymbol{q}}_i + \sum_k \sum_i \sum_j [\boldsymbol{q}_i^T C_{ijk} \dot{\boldsymbol{q}}_j] \boldsymbol{e}_k$

- A nonlinear local connecting function must also satisfy

  - $\boldsymbol{\Phi} = \boldsymbol{x}$

  - $\dfrac{d\boldsymbol{\Phi}}{dt} = \dot{\boldsymbol{x}}$

  - $\dfrac{d^2\boldsymbol{\Phi}}{dt^2} = \boldsymbol{f}(\boldsymbol{\Phi})$

Additional questions:

- How did you prove this?
- What about non-conservative systems?
- What assumptions did you make about $\boldsymbol{f}$ and $\boldsymbol{F}$?
- Did you assume a form for $\boldsymbol{F}$?
- What is N? How do you determine this number?
- Is this the only admissible form for $\boldsymbol{F}$ or one of many possible choices?

The first step to identify nonlinear connecting functions is to assemble a system of algebraic equations for connecting functions.

- Remember: a nonlinear connecting function must satisfy

  - $\mathbf{\Phi} = \mathbf{x},$ $\qquad\qquad \dfrac{d\mathbf{\Phi}}{dt} = \dot{\mathbf{x}},$ and $\qquad \dfrac{d^2\mathbf{\Phi}}{dt^2} = \mathbf{f}(\mathbf{\Phi})$

- We pick nonlinear modes as $\mathbf{q}$'s, i.e.
  $$\mathbf{q}_i(t) = \boldsymbol{\gamma}_i + \textstyle\sum_j\left[\boldsymbol{\alpha}_j(T_i)cos\left(j\frac{2\pi}{T_i}t\right) + \boldsymbol{\beta}_j(T_i)\sin\left(j\frac{2\pi}{T_i}t\right)\right]$$

- Update $\mathbf{\Phi}$ with $\mathbf{q}$'s, i.e.
  $$\mathbf{\Phi} = \mathbf{d}(\boldsymbol{\tau}) + \textstyle\sum_i A_i(\boldsymbol{\tau})\mathbf{q}_i + \sum_i B_i(\boldsymbol{\tau})\dot{\mathbf{q}}_i + \sum_k\sum_i\sum_j[\mathbf{q}_i^T C_{ijk}(\boldsymbol{\tau})\dot{\mathbf{q}}_j]\mathbf{e}_k$$

- Substitute $\mathbf{\Phi}$ in the necessary condition, i.e.
  $$\mathbf{g}(\mathbf{\Phi}, \mathbf{x}, \dot{\mathbf{x}}) = \mathbf{0}$$

- Given an arbitrary set of initial condition $\mathbf{x}(0) = \mathbf{u}, \dot{\mathbf{x}}(0) = \mathbf{v}$, then we discretize time, i.e. $t = 0, t_1, \ldots, t_k$ and
  $$\mathbf{x}(t_k) = \mathbf{u} + \mathbf{v}t_k + \frac{1}{2}\mathbf{f}(\mathbf{u})t_k^2 + \frac{1}{3!}\mathbb{J}_{\mathbf{x}}^{\mathbf{f}}(\mathbf{u})\mathbf{v}t_k^3 + \mathrm{O}\{t_k^4\},$$

- This results in a set of nonlinear algebraic equations
  $$\mathbf{g}(\mathbf{d}, A_i, B_i, c_{ijk}, \boldsymbol{\tau}, \mathbf{u}, \mathbf{v}) = \mathbf{0}$$

One approach to solve the nonlinear system of algebraic equations is Homotopy Analysis.

- Remember $\mathbf{g}(\mathbf{d}, A_i, B_i, c_{ijk}, \boldsymbol{\tau}, \mathbf{u}, \mathbf{v})$

  - is a nonlinear algebraic system of equations: it requires an initial guess.

    - A solution is a connecting function identified by a

      - a set of coefficients, and

$$\Phi_0 = \Phi_0\big(\mathbf{d}, A^n_{i=1}, B^n_{i=1}, C^{n,n}_{i,j=1,k=1}, \boldsymbol{\tau}\big)$$

      - a vector of periods of set of periodic solutions $\boldsymbol{q}_i(t)$

    - Also note, a solution is also a solution of the original system defined by $\boldsymbol{f}$.

- On e method, called Homotopy analysis, defines a Homotopy relation (path) between the known and unknown solutions of the system.

An example of Homotopy Relation (path)

Legend:
- H(x,s=0)sin(πx)
- H(x,s=0.2)
- H(x,s=0.6)
- H(x,s=0.9)
- H(x,s=1)=2x^2(x-1)

Axis labels: H(x,s) vs x
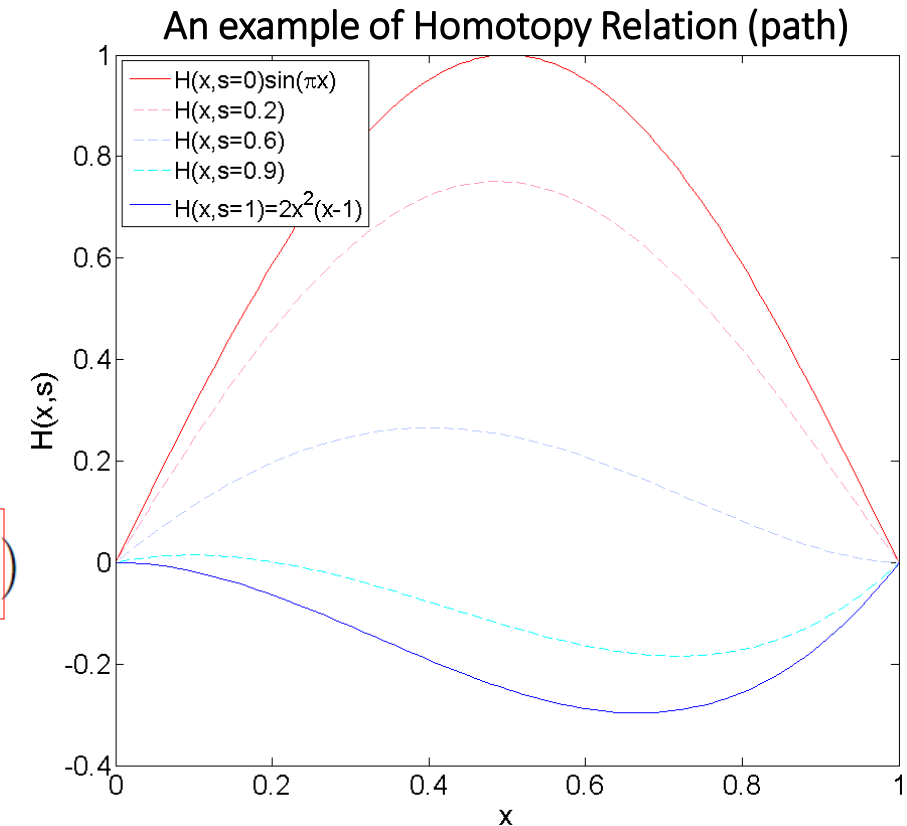
Homotopy paths must be defined! Here's an example for a 2DOF system.

- The Homotopy path

$$\bar{u} = \{\bar{u}(r)|\bar{u}(0) = u_0,\; \frac{d}{dr}E(\bar{u}) \neq 0\}$$

$$\bar{v} = 0$$

$$s = \frac{E(\bar{u}) - E(u_0)}{E(u) - E(u_0)} \in [0,1],\; E(u_0) \to 0$$

$$\bar{\tau} = \{\bar{\tau}(s)|\bar{\tau}(0) = \tau_0\}.$$
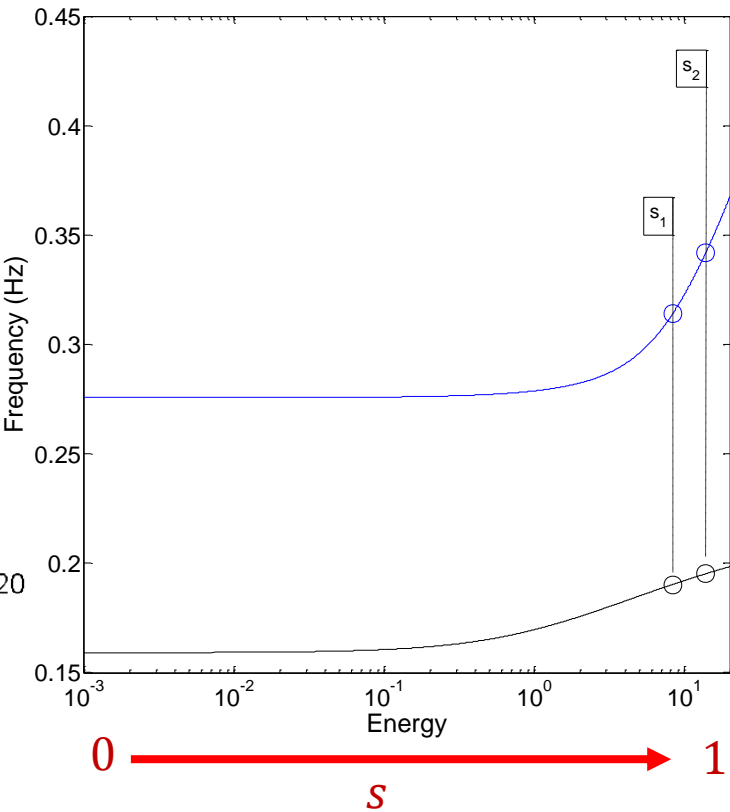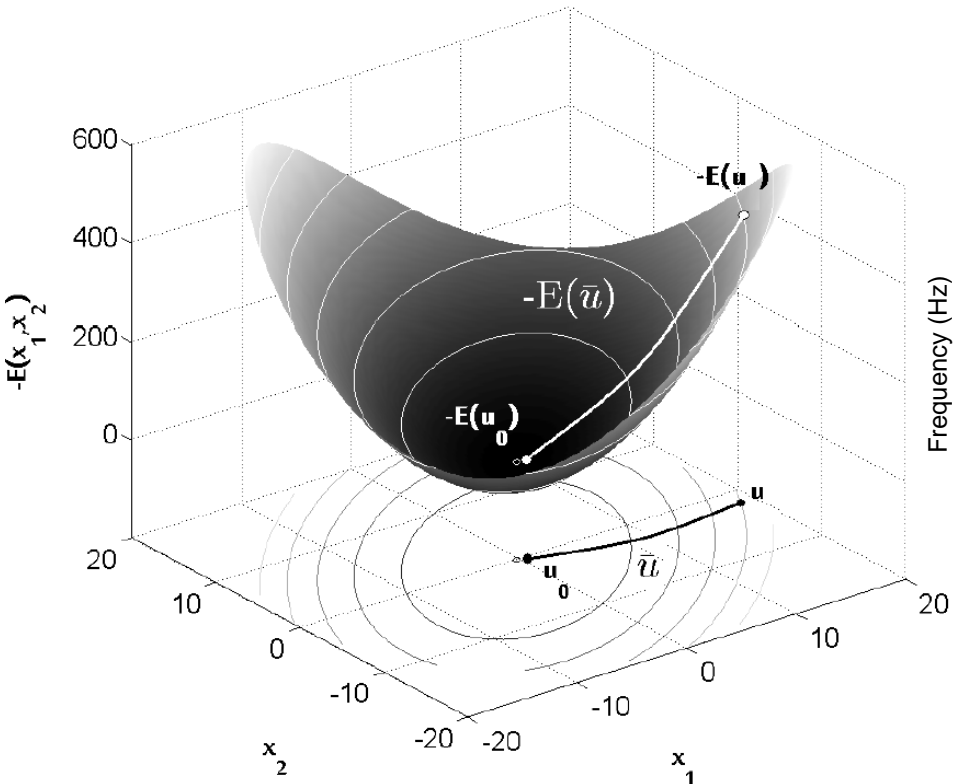
- The known solutions is linear connecting function of linear modes of the system at a point very close to the fixed point of the system.
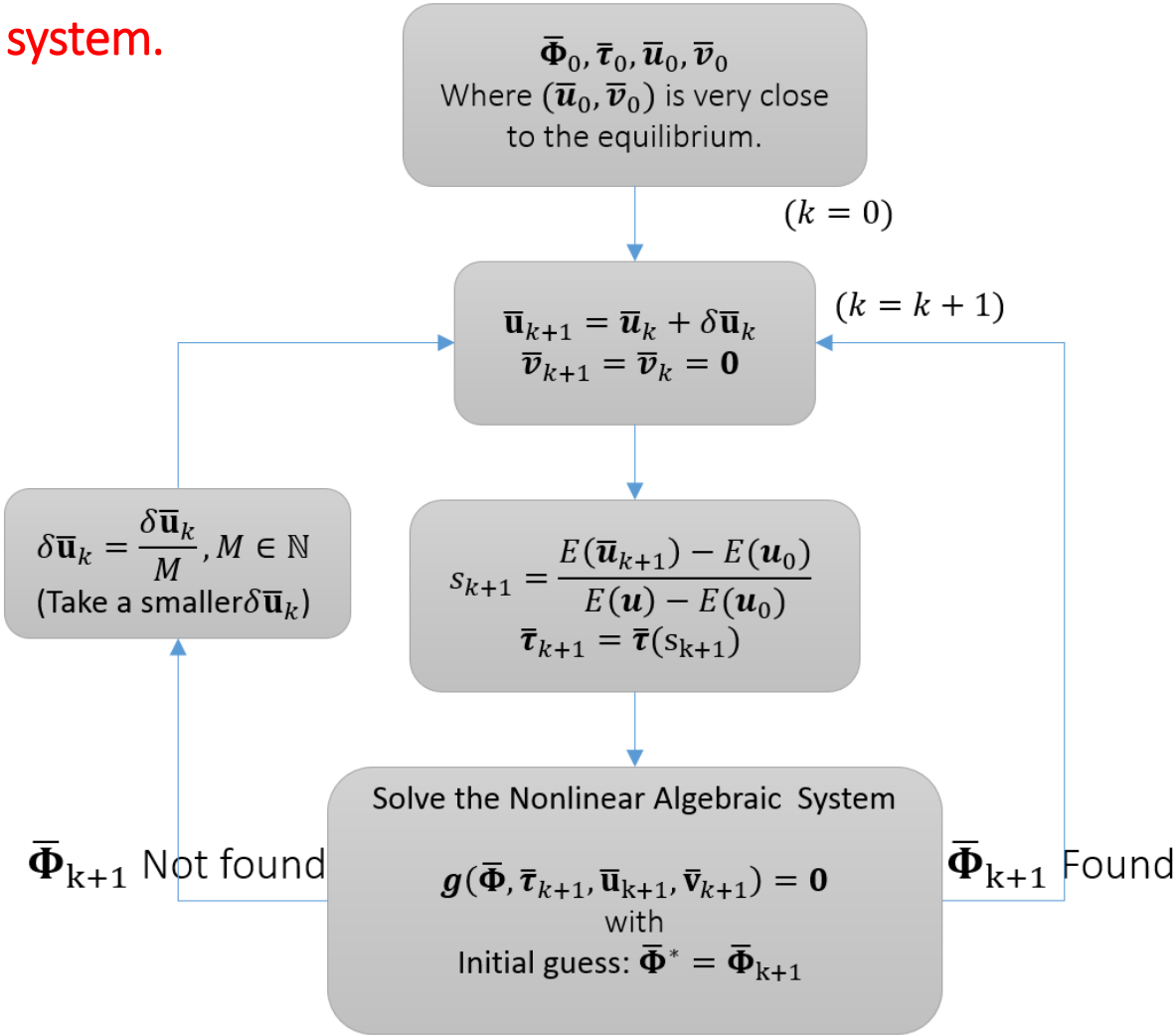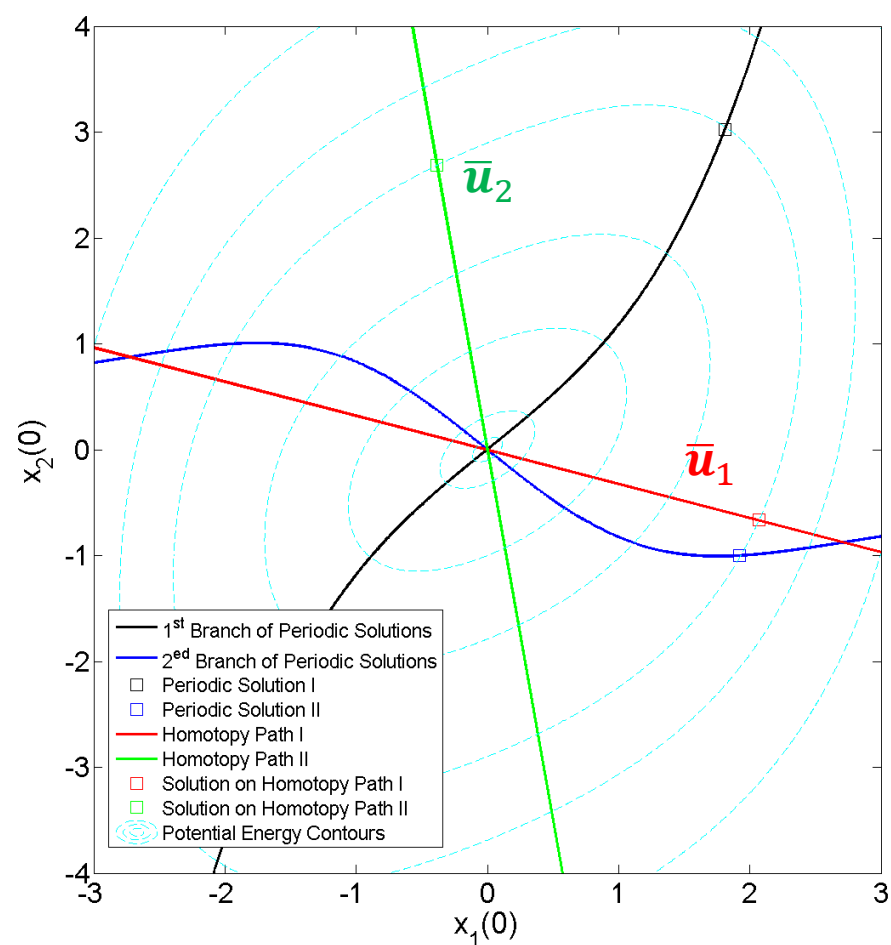
$$A_i, = \bar{a}_i I_n$$

$$\tau = \tau_0$$

$$d = 0,\; B_i = 0_{n \times n},\; C_{ijk} = 0_{n \times n}$$

Two examples of Homotopy paths.

- Two Homotopy paths were defined as straight lines, which cross equipotential contours transversally, starting from two points $\boldsymbol{u}_0^{(1)}$ and $\boldsymbol{u}_0^{(2)}$ very close to the fixed point of the system.

Two examples of Homotopy paths.

- Homotopy paths can be used to find connecting functions at any arbitrary point in the state space.

- Drawback: one must be able to define a Homotopy path and a monotonic Homotopy parameter $s: 0 \rightarrow 1$.

- It's computationally convenient. One can to find a solution (connection function) at any point using periodic solutions with the same energy.

- The results show consistent accuracy even in significantly nonlinear region. The frequencies of the first and the second can rise up to 36% and 145% compared to their linear natural frequencies.
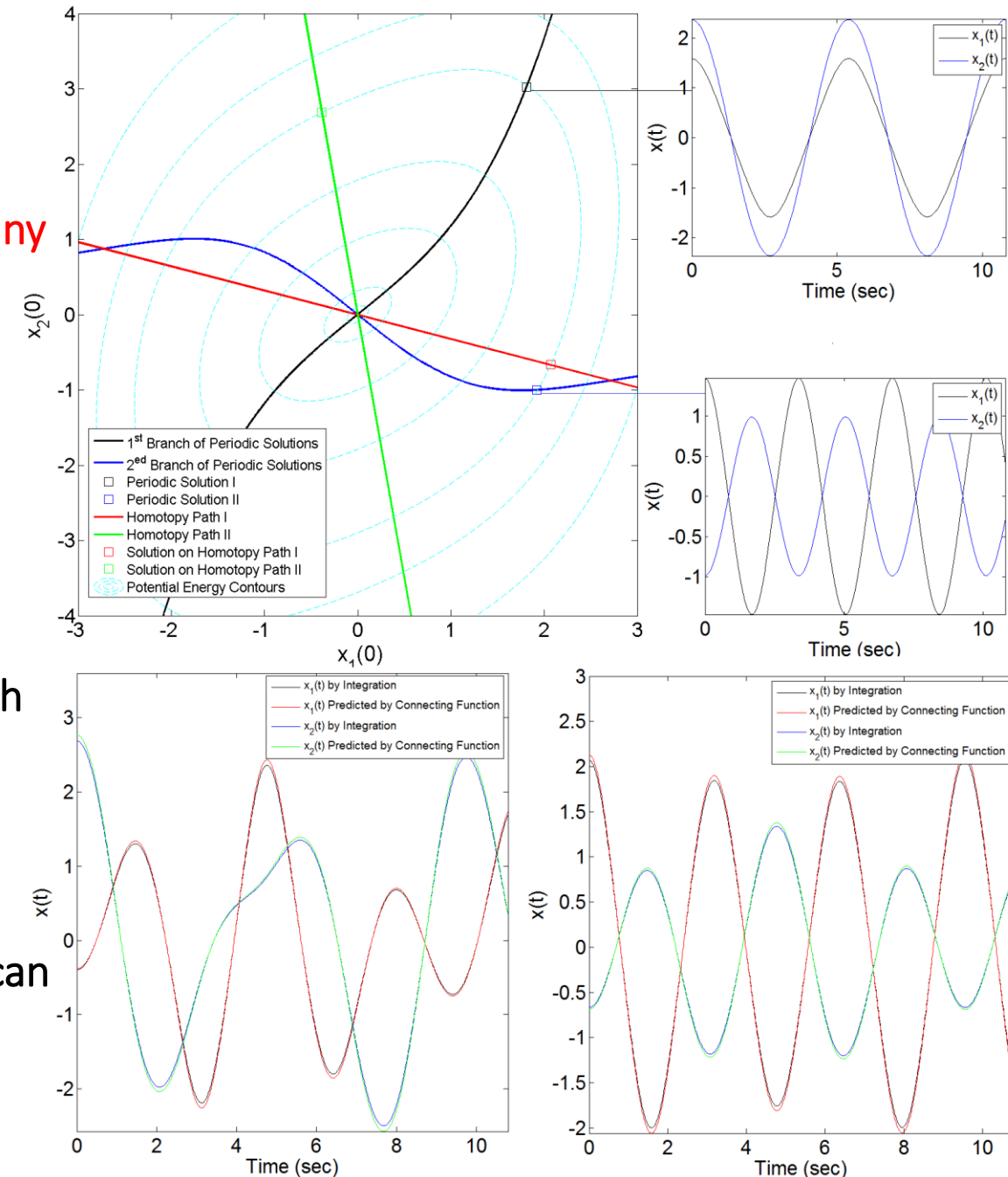
A second approach to solve the system of algebraic equations is to use continuation.
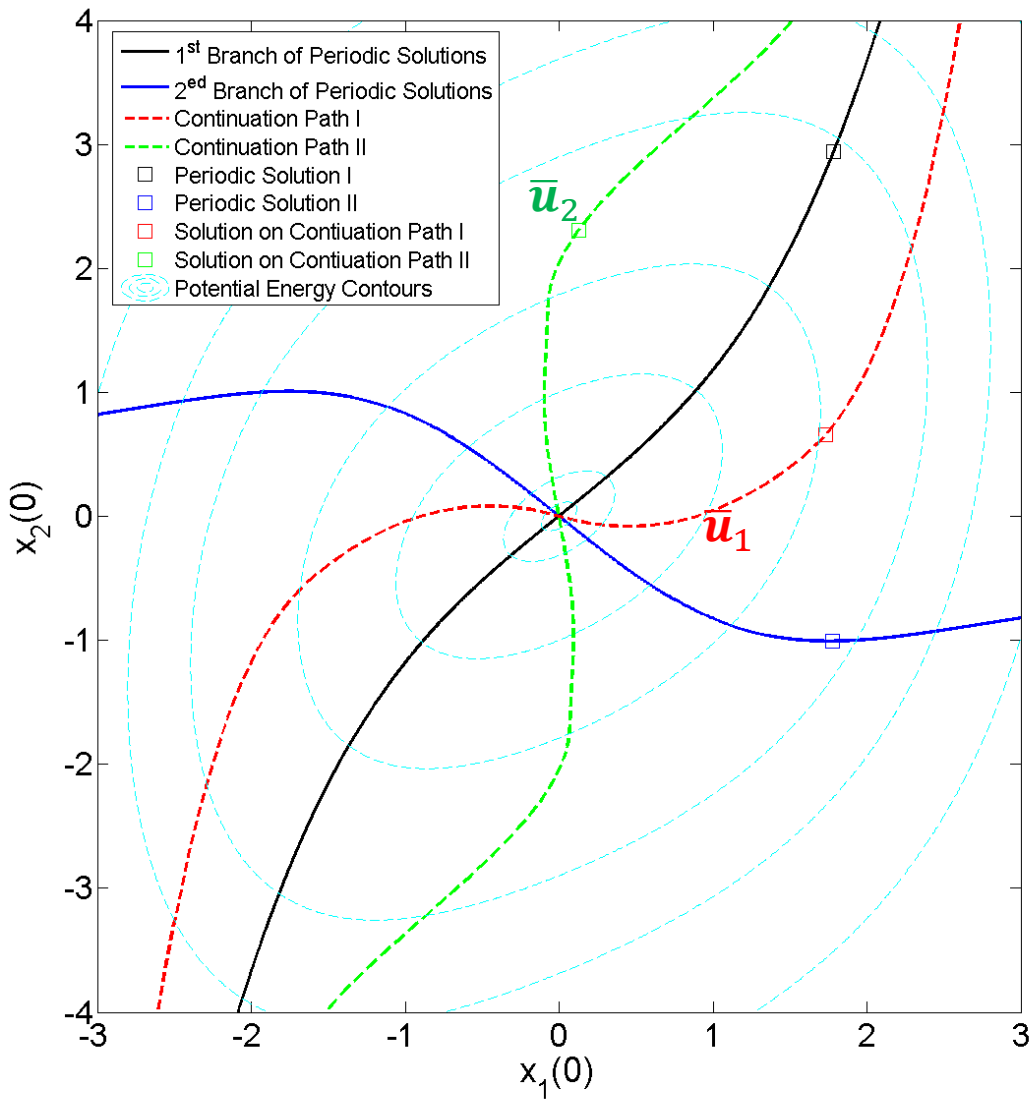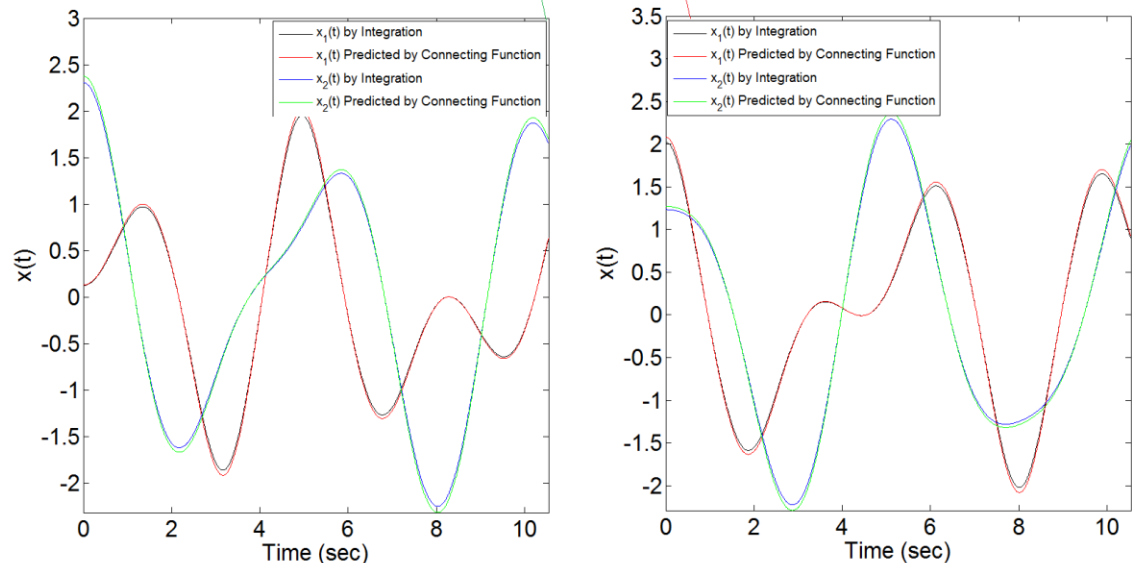

- Two continuation paths were obtained, starting from the same two points $u_0^{(1)}$ and $u_0^{(2)}$ very close to the fixed point of the system.
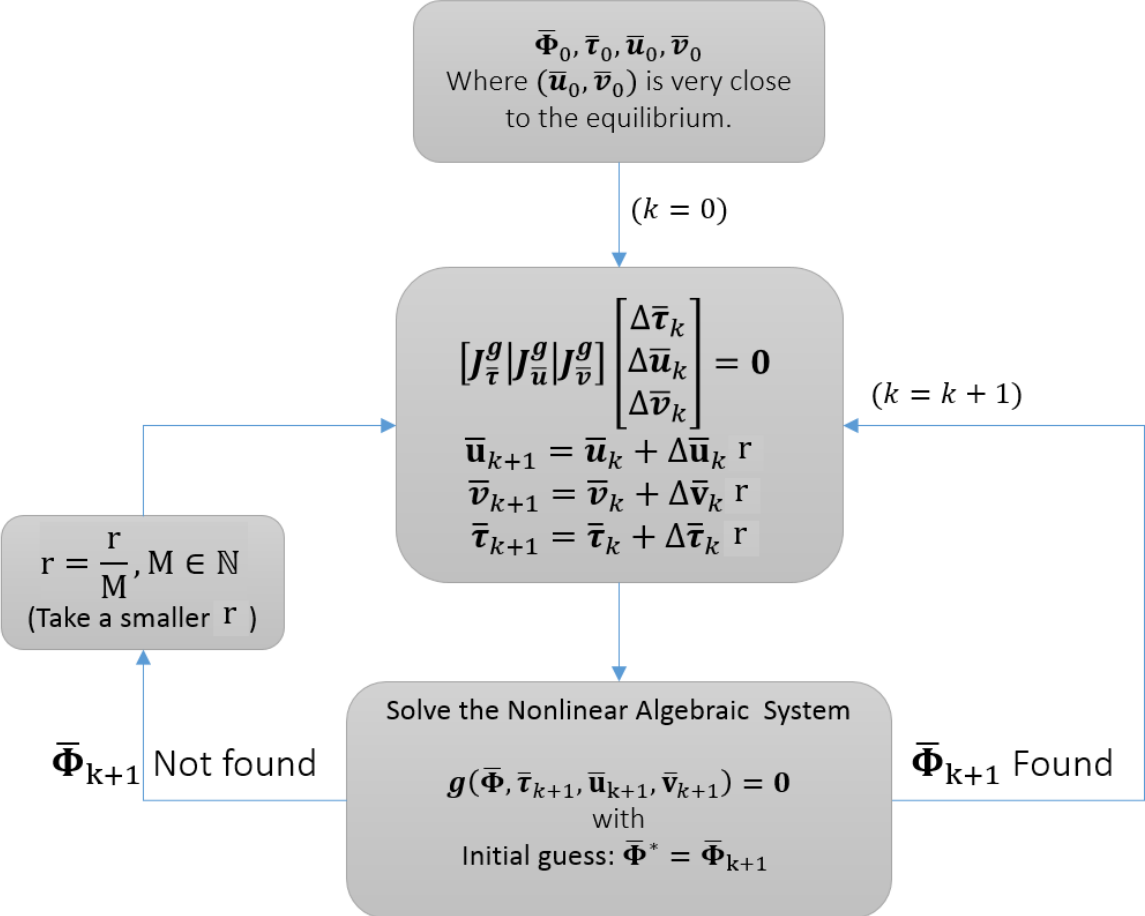
Two examples of continuation paths.

- Continuation paths are determined by the dynamics of J_τ^g, J_u^g and J_v^g and the continuation parameter (step-size), i.e. the destination of continuation paths cannot be set in advance.
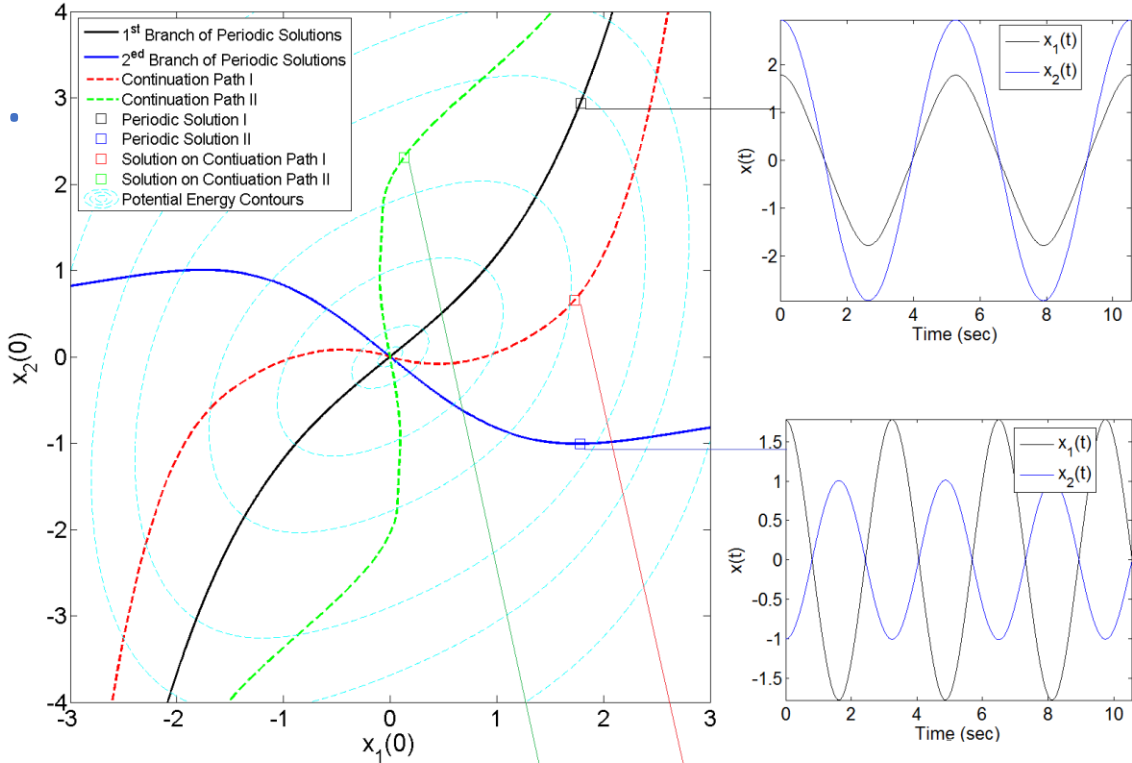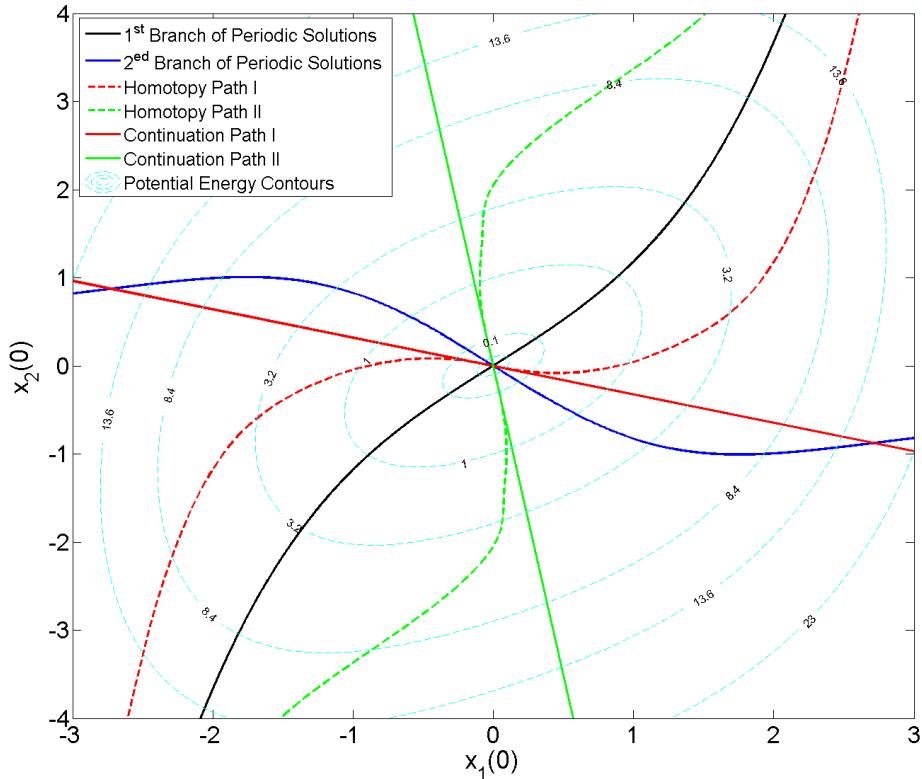
- Continuation paths follow the Homotopy paths very closely in the linear region!

A third approach is also provided which does not require starting from a known solution in the linear region!

- This approach calculates a linear approximate connecting function (of a set of almost orthogonal periodic solutions) and then uses it as the initial guess for solving the nonlinear algebraic equations.
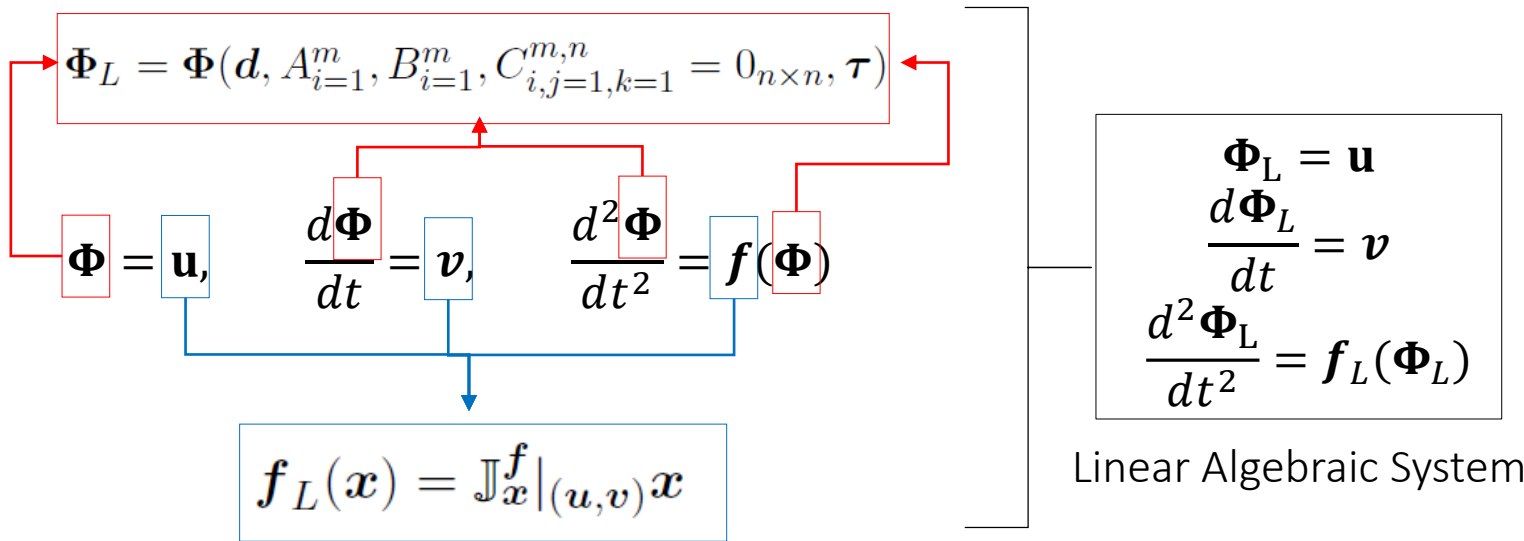
$$\boldsymbol{\Phi}_L = \boldsymbol{\Phi}(\boldsymbol{d}, A_{i=1}^m, B_{i=1}^m, C_{i,j=1,k=1}^{m,n} = 0_{n\times n}, \boldsymbol{\tau})$$

$$\boldsymbol{\Phi} = \mathbf{u}, \qquad \frac{d\boldsymbol{\Phi}}{dt} = \boldsymbol{v}, \qquad \frac{d^2\boldsymbol{\Phi}}{dt^2} = \boldsymbol{f}(\boldsymbol{\Phi})$$

$$\boldsymbol{f}_L(\boldsymbol{x}) = \mathbb{J}_{\boldsymbol{x}}^{\boldsymbol{f}}\big|_{(\boldsymbol{u},\boldsymbol{v})}\boldsymbol{x}$$

$$\boldsymbol{\Phi}_L = \mathbf{u}$$
$$\frac{d\boldsymbol{\Phi}_L}{dt} = \boldsymbol{v}$$
$$\frac{d^2\boldsymbol{\Phi}_L}{dt^2} = \boldsymbol{f}_L(\boldsymbol{\Phi}_L)$$

Linear Algebraic System

- In order to avoid solving the linear system, this approach uses almost orthogonal periodic solutions.

Two functions are almost orthogonal if their projections on a finite orthogonal basis is orthogonal!

$$x_1(t) = \sum_{m\in\mathbb{N}} A_{1m}\cos(m\omega_1 t)$$

$$x_2(t) = \sum_{m\in\mathbb{N}} A_{2m}\cos(m\omega_2 t)$$

$$S = \{\cos(\omega_0 t), \dots \cos(5\omega_0 t)\}$$

$$[A_{11}\dots A_{15}], \qquad [A_{21}\dots A_{25}]$$

$$\frac{\omega_1}{\omega_2} = \frac{p_1\omega_0}{p_2\omega_0} = \frac{p_1}{p_2}$$

$$p_1, p_2 > 5 \quad \rightarrow \quad [A_{11}\dots A_{15}][A_{21}\dots A_{25}]^{\mathrm{T}} = 0$$

The linear approximation to the connecting function provides a good initial guess for solving the nonlinear algebraic system even in the nonlinear region.

- The first and the second modes show 19% and 35% increase in their frequencies compared to the first and second linear natural frequencies.

Additional questions:

- How do you find the set of almost orthogonal periodic solutions?
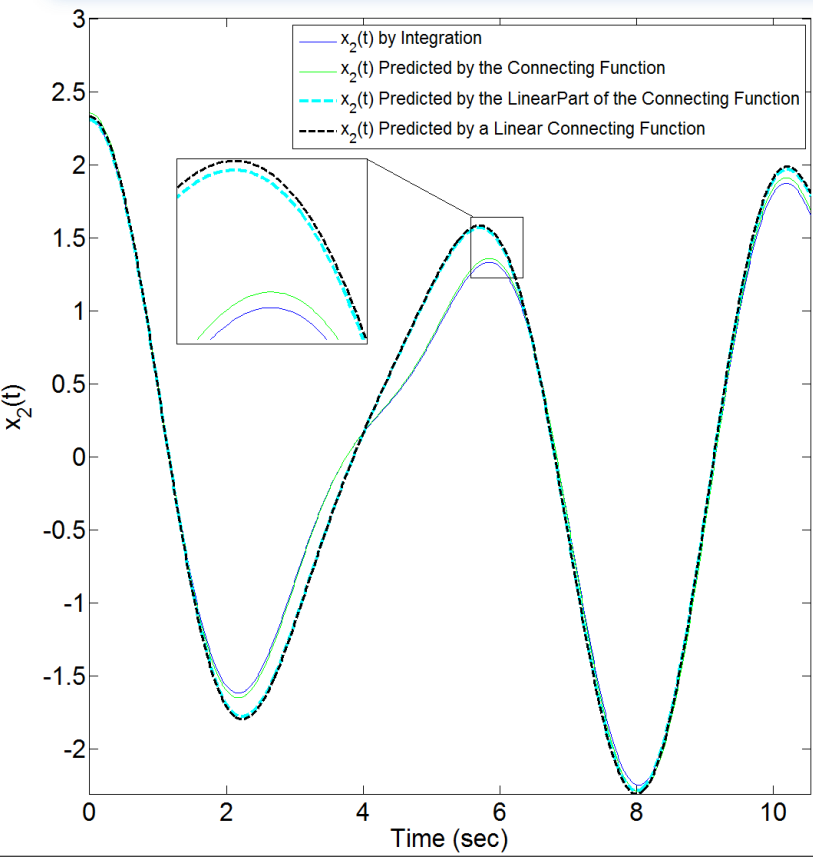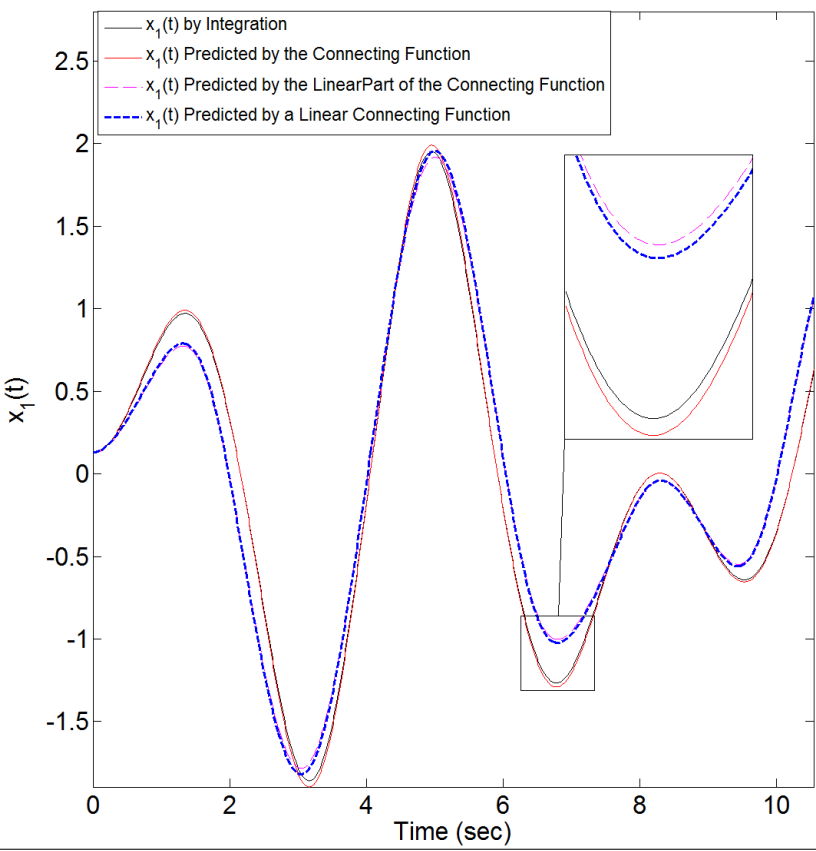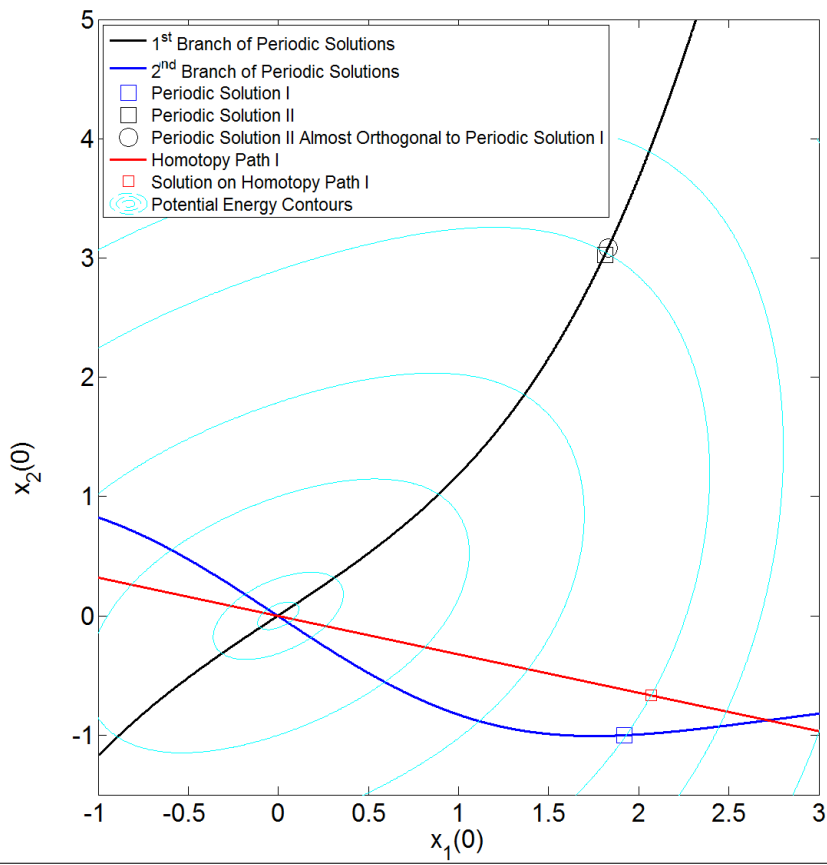
This work has provided a new definition and new methods of calculation and accurate stability analysis of nonlinear modes.

- A new definition for a nonlinear mode of, specifically, Instantaneous Center Manifold (ICM) was proposed.
  - I.   The proposed definition encompasses all the previous nonlinear mode definitions.
  - II.   It also leads to new methods of calculation of nonlinear modes that do not require any previously known solution as an initial guess.

- An extremely simple yet effective method, i.e. Multi-harmonic Multiple-point Collocation (MMC), for finding periodic solutions of conservative nonlinear systems was presented.
  - I.   MMC uses a similar condition to the one used in multiple-point shooting methods, however, it does not require integration of the vector field over any period of time.
  - II.   MMC is capable of finding more than one periodic solutions of the nonlinear system in each solution.
  - III.   Although not presented in this document, the algorithm has been implemented in a continuation framework and seems to be more computationally efficient than shooting algorithms.

- A set of limits for two sources of error in the process of calculating Floquet exponents is presented.
  - I.   These limits were used to propose a criteria for validation of near-zero Floquet exponents.
  - II.   An algorithm was also proposed which adjusted the integration tolerance and approximation level for the periodic solutions using the criteria, to assure that valid Floquet exponents were calculated along an entire branch.
  - III.   The proposed algorithm also reduces the computational cost of finding periodic solutions by providing an adaptive tolerance for the periodic orbit solver.

This work has presented the general form and three methods of calculation of nonlinear connecting functions.

- The general form of nonlinear connecting functions is provided.

- The concept of almost orthogonality and order of almost orthogonality is defined for periodic solutions.

    - A numerical scheme for finding a set of almost orthogonal periodic solution (of any order) is provided.

- Three numerical approaches of calculation of connecting functions are provided.

    I. The first two approaches use Homotopy analysis and continuation respectively to identify continuous branches of connecting functions.

        I. However both methods demand starting from a known connecting function.

    II. The third method, however, overcomes this constraint by finding a linear approximation of a connecting function at any arbitrary point in the state space and using it as the starting point in the search for the nonlinear connecting function.

        I. The concept of almost orthogonality makes it possible for the third approach to efficiently find the a linear approximation of connecting functions at any arbitrary point regardless of its proximity to the equilibrium of the system

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Thanks!

Questions?