NIFO and NIXO Estimators for SDOF Systems: Impact of the Output Terms Choice on the Results

Michael Kwarta and Matthew S. Allen

Contents

Abstract

This report proposes a strategy on how to utilize NIXO approaches in the black-box identification of a single degree of freedom mechanical system. The methods first use a polynomials of various degrees to distinguish which nonlinear terms are dominant in the systems response. After detecting that the cubic nonlinearity is the most dominant an additional analysis of an impact of a single extra nonlinear term on the estimation is conducted on. Results obtained using NIXO and NIFO methods are presented and compared to each other.

Keywords: Nonlinear System Identification, NIFO methods, NIXO methods, Nonlinear H-Estimators, Black-Box methods

1 Introduction

NIFO and NIXO are the nonlinear identification algorithms which allow for estimation of the frequency response function (FRF) of the underlying linear system as well as the parameters defining the nonlinearity. NIFO methods treat the nonlinearities as feedback forcing terms, while the system identification using NIXO takes place via augmenting the number of the outputs. Thus, the abbreviations of the methods (which were first introduced in [1, 2] and [3], respectively) come from:

– NIFO: Nonlinear Identification through Feedback ot the Outputs

– NIXO: Nonlinear Identification through eXtended Outputs

In this work we would like to analyze an impact of the output terms choice on the estimators accuracy. The methods are evaluated numerically using a single degree of freedom system with a cubic nonlinear term. In the identification process – nonlinearity is assumed to be represented as a polynomial of various degrees. The results obtained using NIFO and NIXO methods are then compared with each other and commented.

2 Case Study – Impact of the Polynomial Degree

Input and output signals used in this case study come from Scenario B104 described in [3]. They are generated using the Duffing equation (1) excited with a swept cosine forcing function. Values describing the mechanical system were proposed in [1] and are given in Tab. 1. Auto– and cross–spectra are obtained by applying 25-seconds-long Hanning windows with 51% of overlapping.

$$m\ddot{x} + c\dot{x} + kx + k_3x^3 = f(t) \tag{1}$$

Tab. 1: Parameters describing SDOF mechanical system with cubic stiffness nonlinearity.

m [kg]	c $\left[\frac{\text{N s}}{\text{m}}\right]$	k $\left[\frac{\text{N}}{\text{m}}\right]$	k_3 $\left[\frac{\text{N}}{\text{m}^3}\right]$
1	4	10^3	10^5

Definition of Swept Cosine Forcing Signal

$$f(t) = F\cos(\Omega(t)\ t) \qquad \Omega(t) = \Omega_{st} + \frac{\Omega_{end} - \Omega_{st}}{t_{end} - t_{st}}(t - t_{st}) \qquad t \in [t_{st},\ t_{end}] \tag{2}$$

Tab. 2: Parameters describing swept cosine forcing function.

Ω_{st} [Hz]	Ω_{end} [Hz]	t_{st} [s]	t_{end} [s]
0.01	15	0	1500

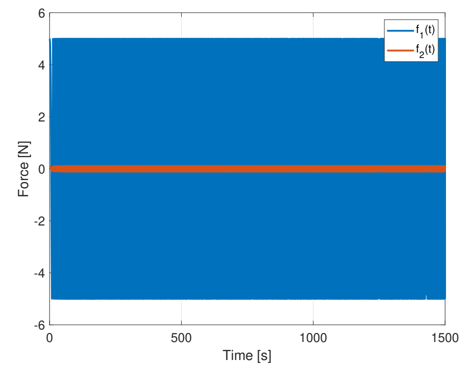
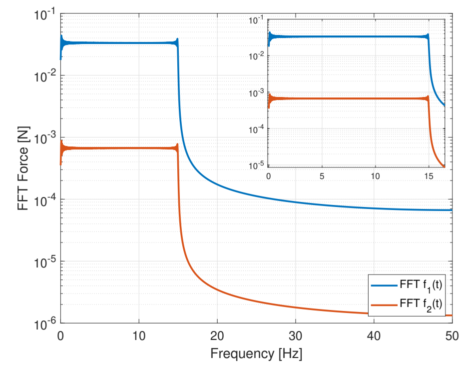
Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B104	101	5	0.01	-	-	-	-	0.3	15

NIFO and NIXO algorithms were tested to estimate the parameters of Eq. (1) summarized in Tab. 1. Model function used in the estimation process is given in Eq. (3). Multiple tests, with different value of polynomial degree p and unknown k_i parameters ($i \in \{2, \ldots, p\}$), were conducted on. The results are presented in Tabs. from 4 to 11. Results are briefly commented at the end of this section.

$$m\ddot{x} + c\dot{x} + kx + k_2 x|x| + k_3 x|x|^2 + \cdots + k_p x|x|^{p-1} = f(t) \quad (3)$$

Tab. 3: SCENARIO B104: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B104	101	5.000	0.100	-	-	-	-	0.3	15

<table>
<tr><th>Signal</th><th>Time domain</th><th>Freq. domain</th></tr>
<tr><td>$f(t)$</td><td>
</td><td>
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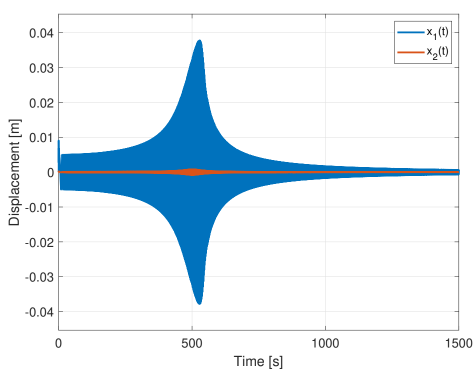
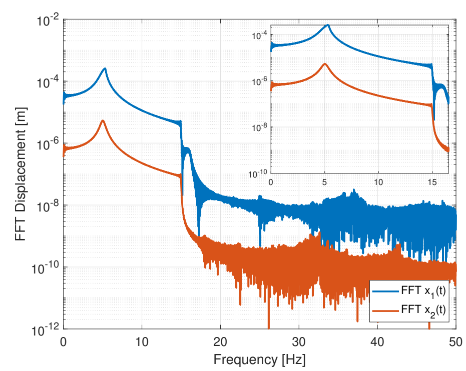
<tr><td>$x(t)$</td><td>
</td><td>
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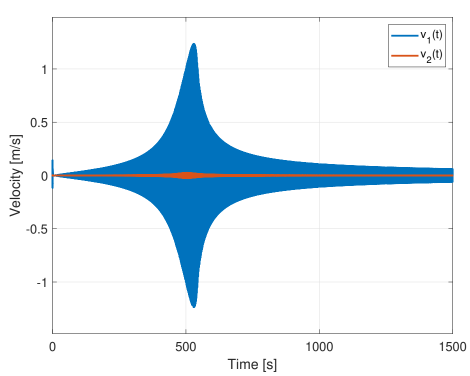
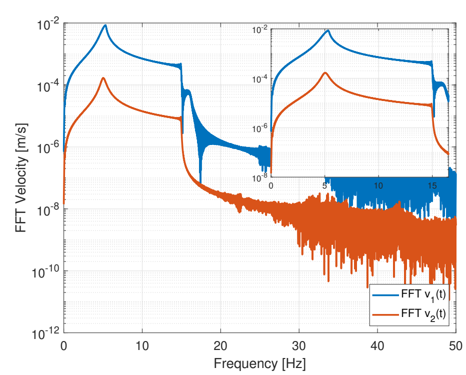
<tr><td>$v(t)$</td><td>
</td><td>
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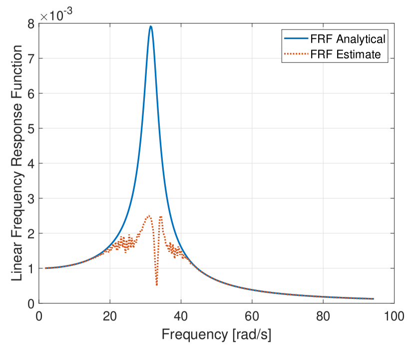
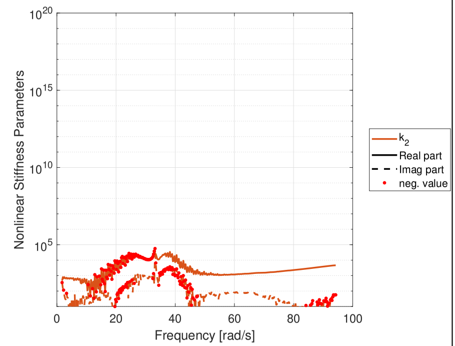
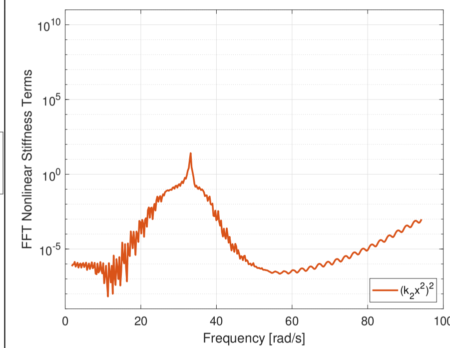
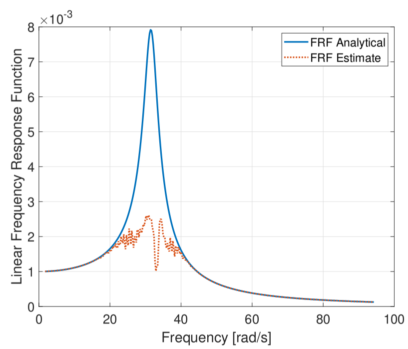
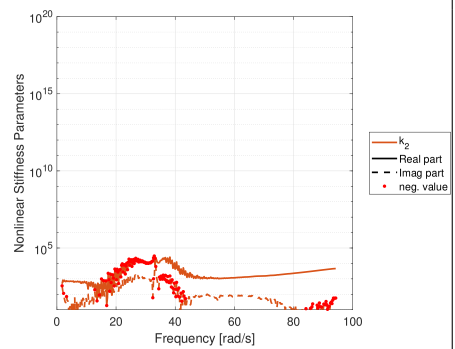
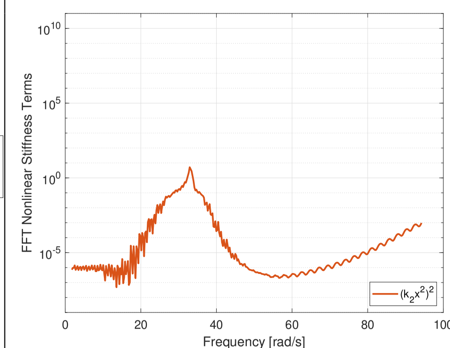
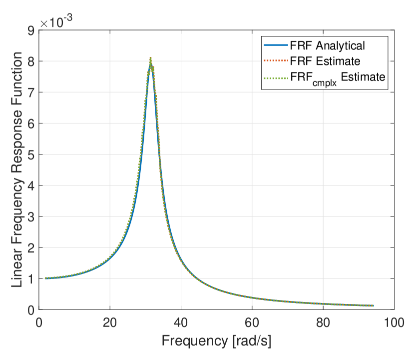
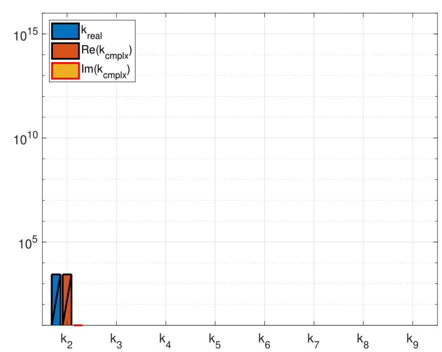
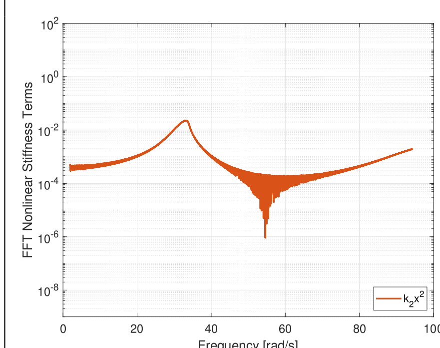
Tab. 4: Degree of stiffness polynomial: 2. Black (red) border around the bars indicates positive (negative) value.

	Linear FRF estimate	Nonlin. param. estimates	Nonlin. terms FFT ($x(t)$ – output signal)
$\mathbf{H}_1$-NIFO			
$\mathbf{H}_2$-NIFO			
$\mathbf{H}_1$-NIXO			

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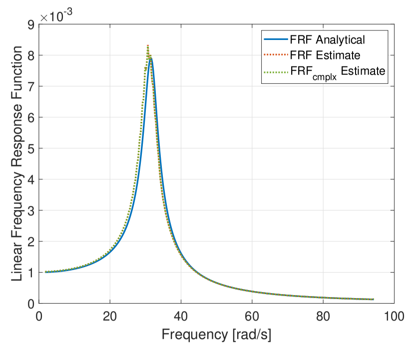
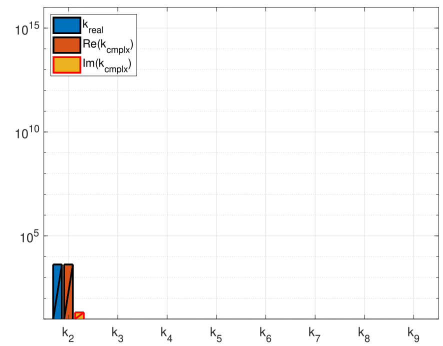
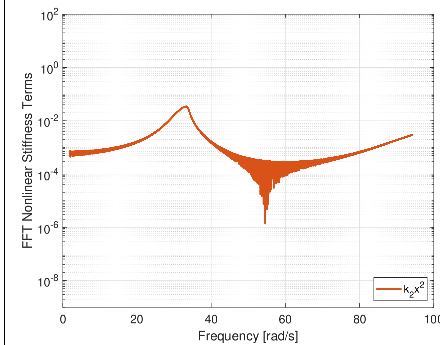
<tr><th></th><th>Linear FRF estimate</th><th>Nonlin. param. estimates</th><th>Nonlin. terms FFT
($x(t)$ – output signal)</th></tr>
<tr><td>H$_2$-NIXO</td><td>
</td><td>
</td><td>
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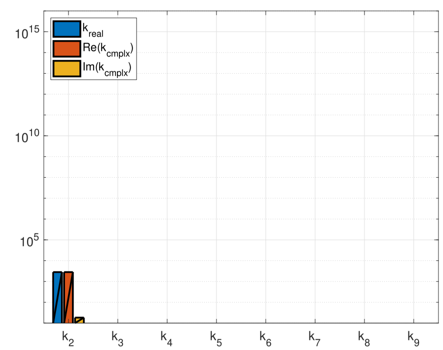
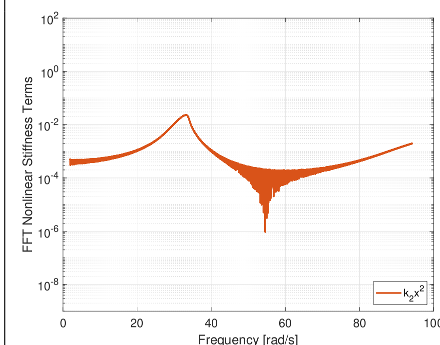
<tr><td>H$_1$-NIXO WLDP</td><td>×</td><td>
</td><td>
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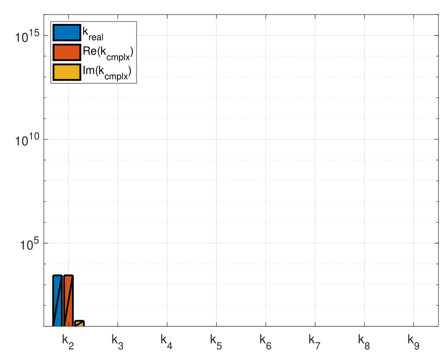
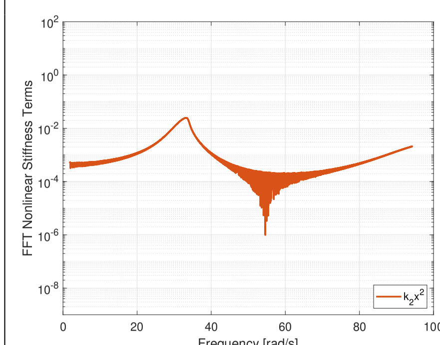
<tr><td>H$_2$-NIXO WLDP</td><td>×</td><td>
</td><td>
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Tab. 5: Degree of stiffness polynomial: 3. Black (red) border around the bars indicates positive (negative) value.

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<tr><th></th><th>Linear FRF estimate</th><th>Nonlin. param. estimates</th><th>Nonlin. terms FFT
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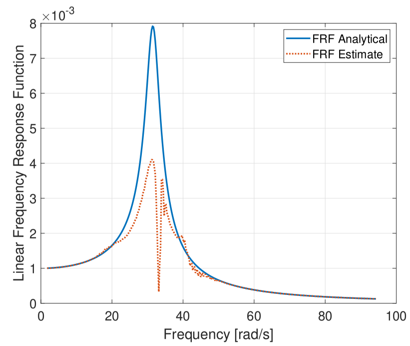
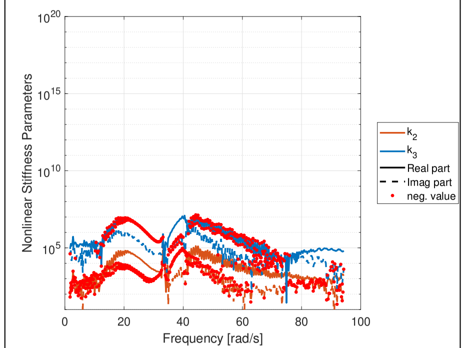
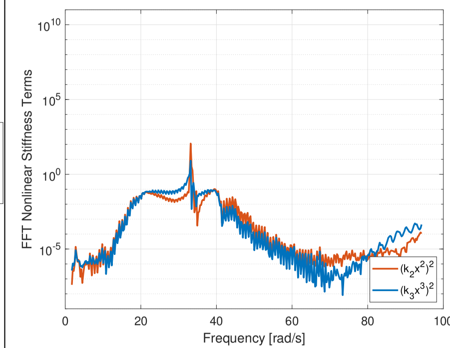
<tr><td>$\mathbf{H}_1$-NIFO</td><td>
</td><td>
</td><td>
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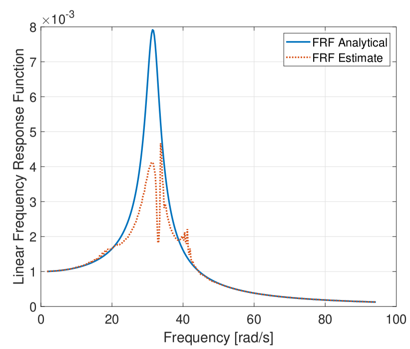
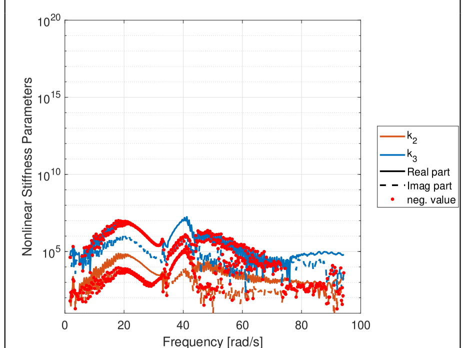
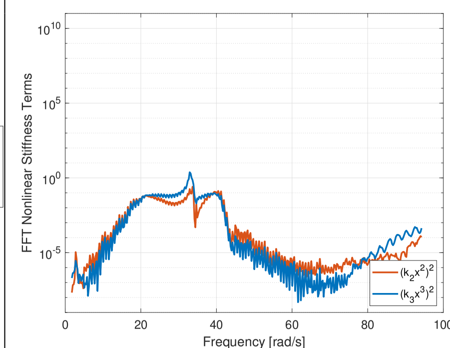
<tr><td>$\mathbf{H}_2$-NIFO</td><td>
</td><td>
</td><td>
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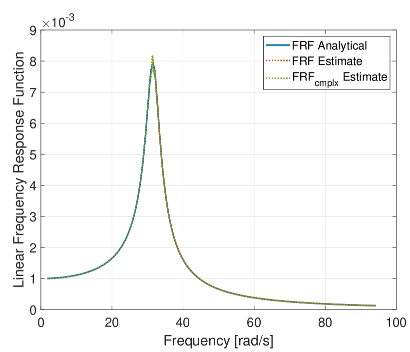
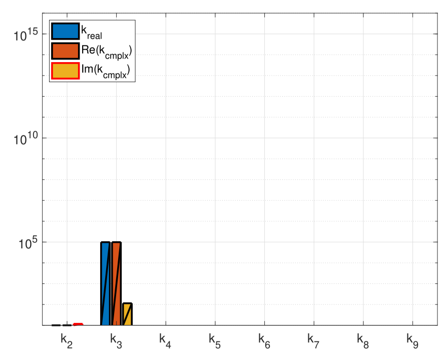
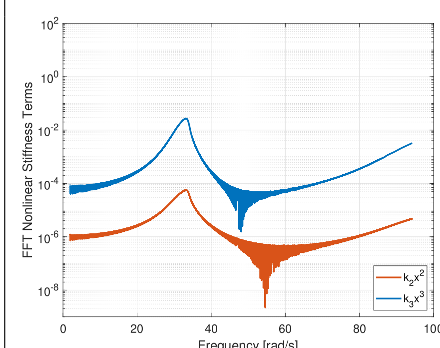
<tr><td>$\mathbf{H}_1$-NIXO</td><td>
</td><td>
</td><td>
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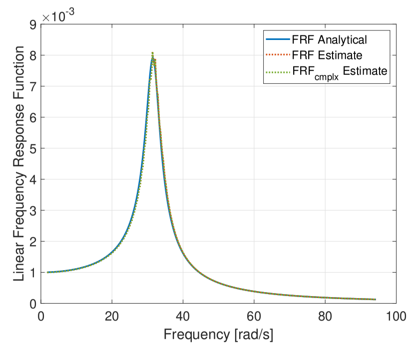
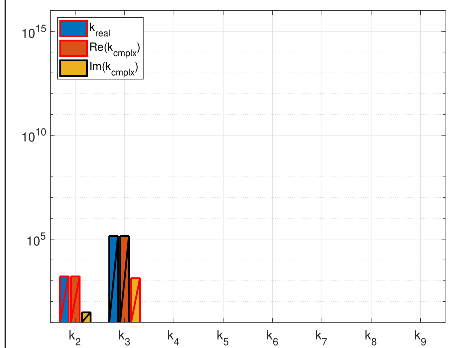
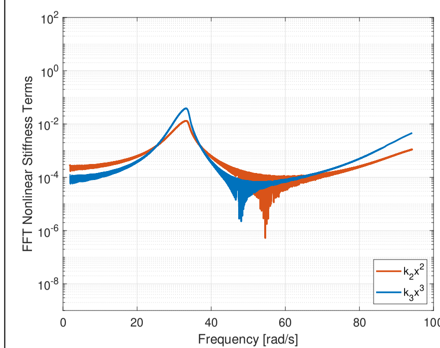
<tr><td>$\mathbf{H}_2$-NIXO</td><td>
</td><td>
</td><td>
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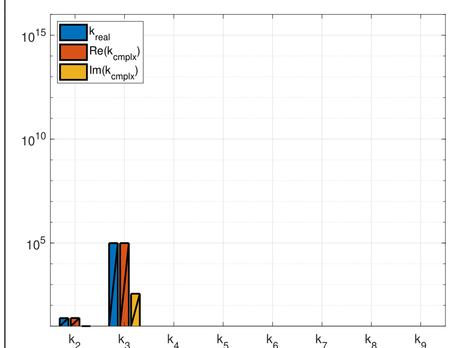
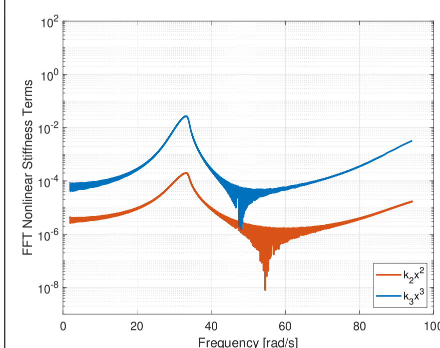
<tr><td>$\mathbf{H}_1$-NIXO WLDP</td><td>×</td><td>
</td><td>
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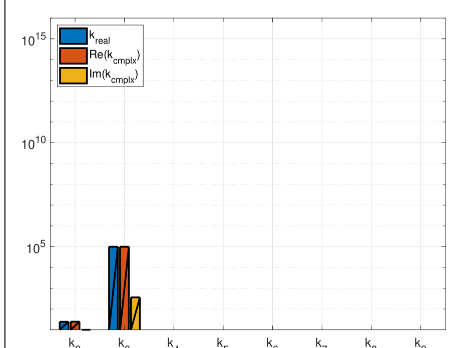
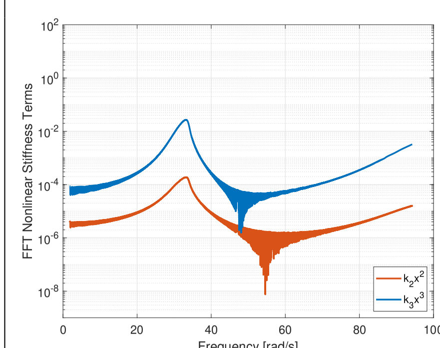
<tr><td>$\mathbf{H}_2$-NIXO WLDP</td><td>×</td><td>
</td><td>
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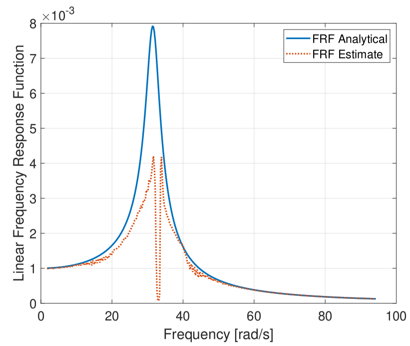
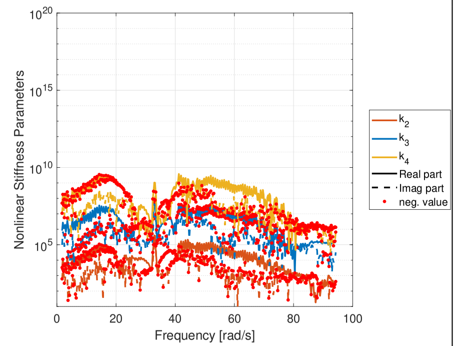
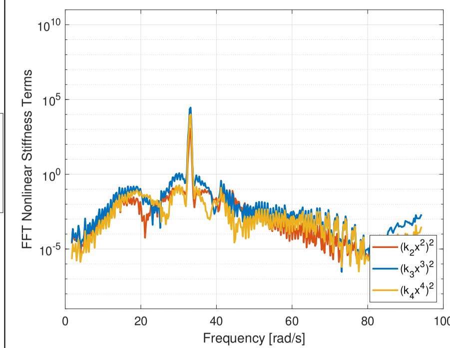
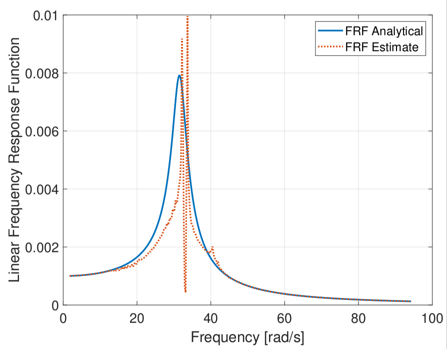
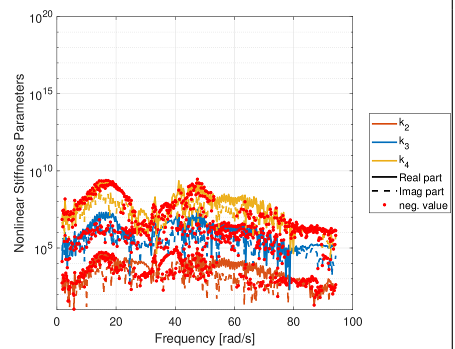
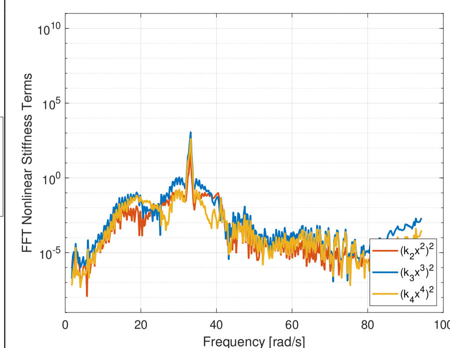
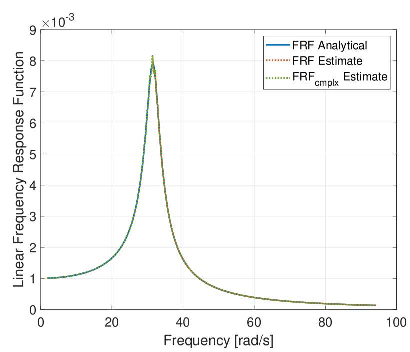
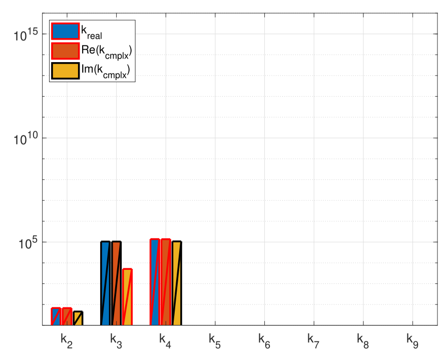
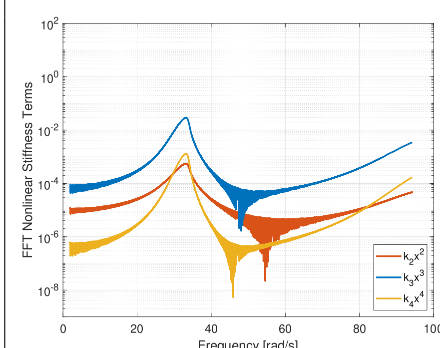
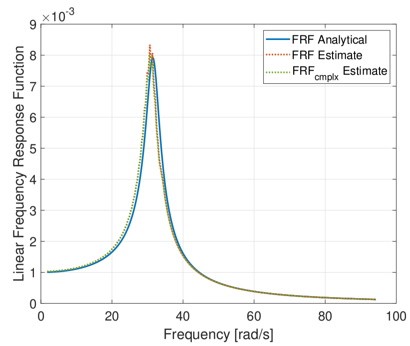
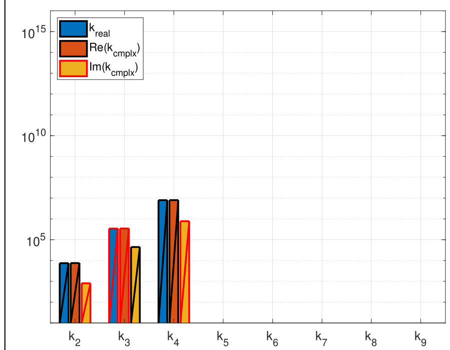
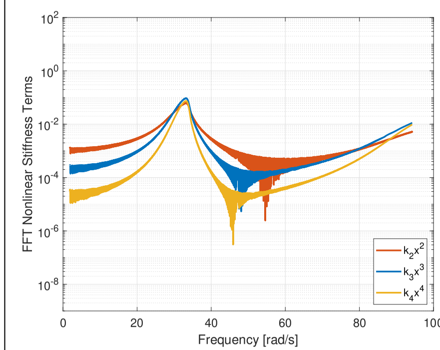
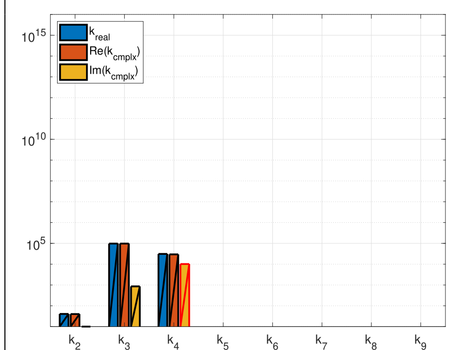
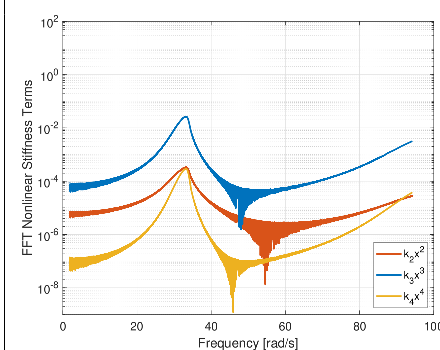
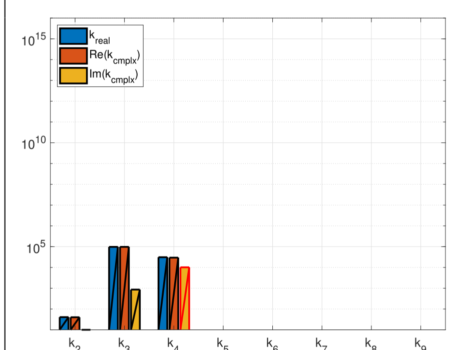
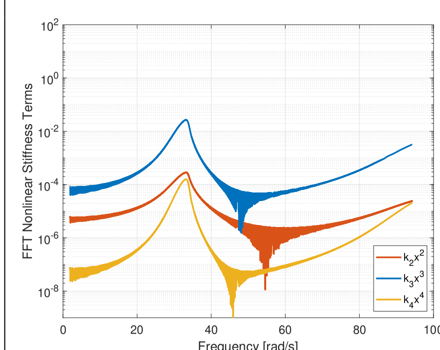
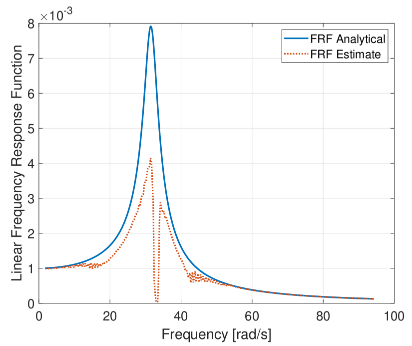
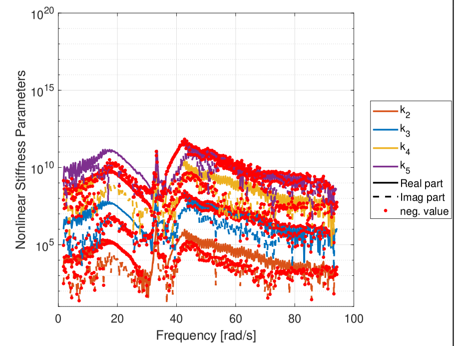
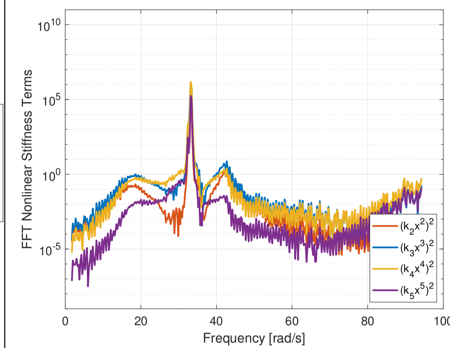
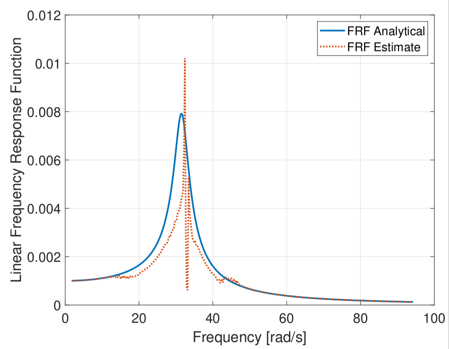
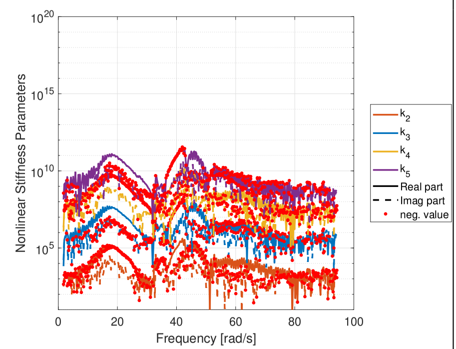
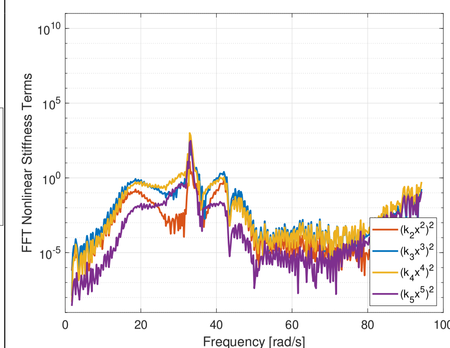
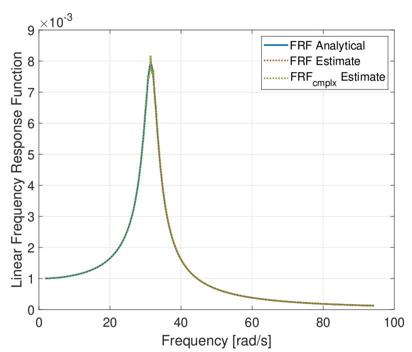
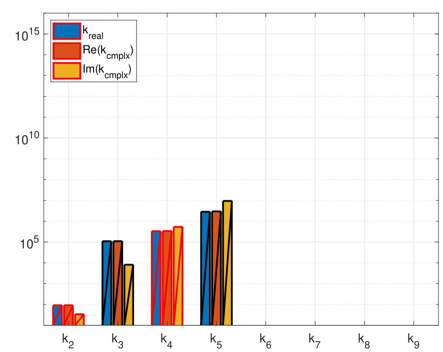
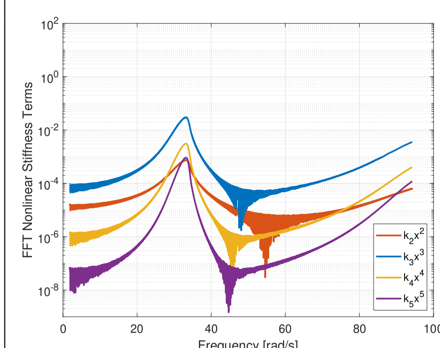
Tab. 6: Degree of stiffness polynomial: 4. Black (red) border around the bars indicates positive (negative) value.

	Linear FRF estimate	Nonlin. param. estimates	Nonlin. terms FFT ($x(t)$ – output signal)
$\mathbf{H}_1$-NIFO			
$\mathbf{H}_2$-NIFO			
$\mathbf{H}_1$-NIXO			

	Linear FRF estimate	Nonlin. param. estimates	Nonlin. terms FFT ($x(t)$ – output signal)
$\mathbf{H}_2$-NIXO			
$\mathbf{H}_1$-NIXO WLDP	×		
$\mathbf{H}_2$-NIXO WLDP	×		

Tab. 7: Degree of stiffness polynomial: 5. Black (red) border around the bars indicates positive (negative) value.

<table>
<tr><th></th><th>Linear FRF estimate</th><th>Nonlin. param. estimates</th><th>Nonlin. terms FFT
($x(t)$ – output signal)</th></tr>
<tr><td>H$_1$-NIFO</td><td></td><td></td><td></td></tr>
<tr><td>H$_2$-NIFO</td><td></td><td></td><td></td></tr>
<tr><td>H$_1$-NIXO</td><td></td><td></td><td></td></tr>
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<table>
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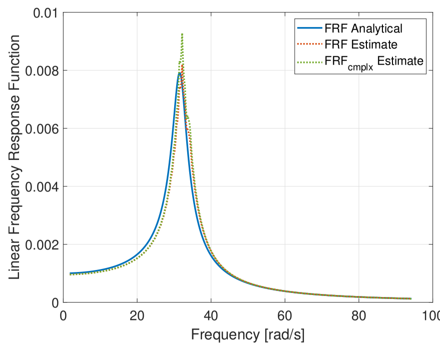
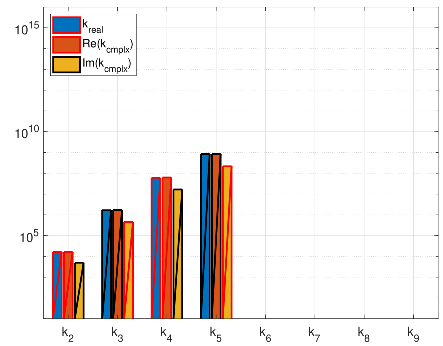
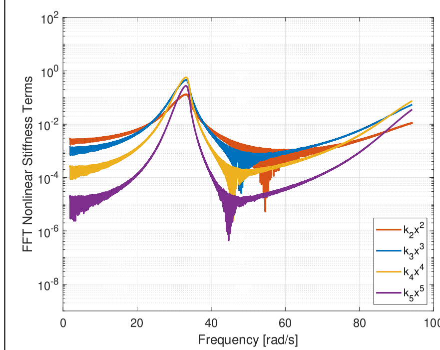
($x(t)$ – output signal)</th></tr>
<tr><td>$\mathbf{H}_2$-NIXO</td><td>
</td><td>
</td><td>
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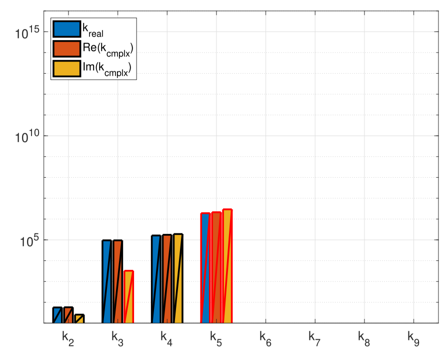
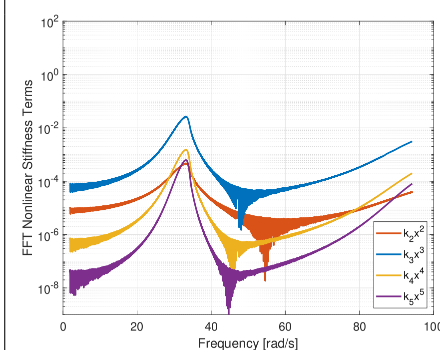
<tr><td>$\mathbf{H}_1$-NIXO WLDP</td><td>×</td><td>
</td><td>
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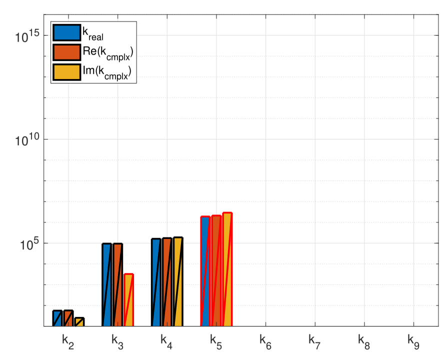
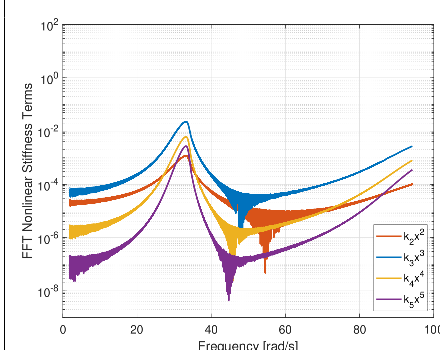
<tr><td>$\mathbf{H}_2$-NIXO WLDP</td><td>×</td><td>
</td><td>
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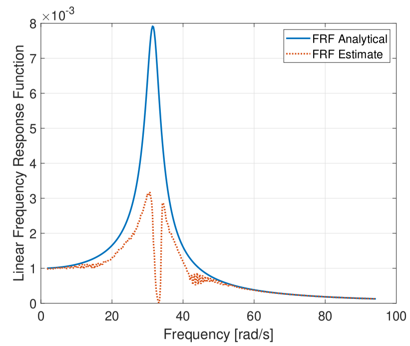
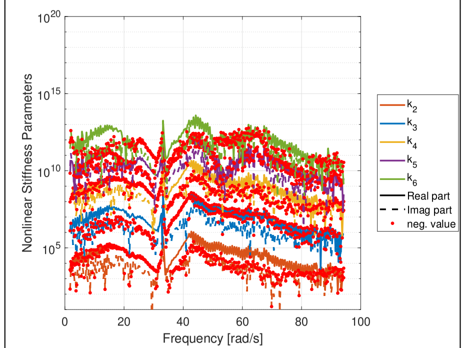
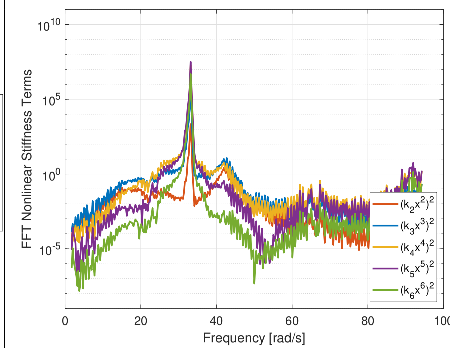
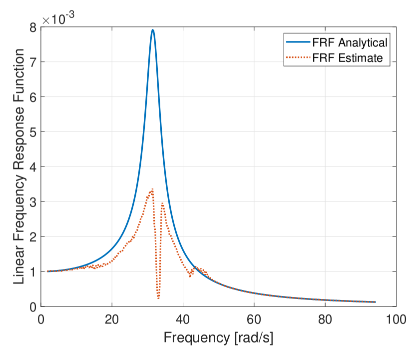
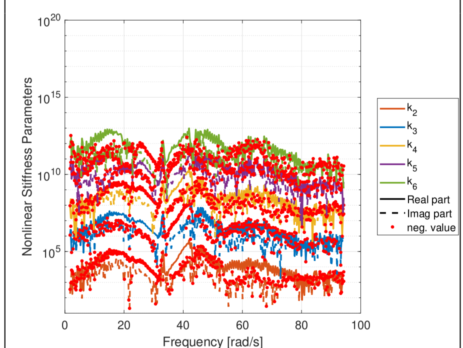
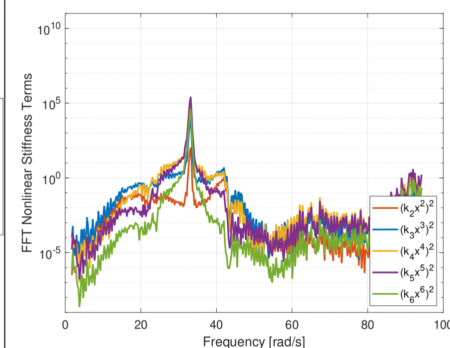
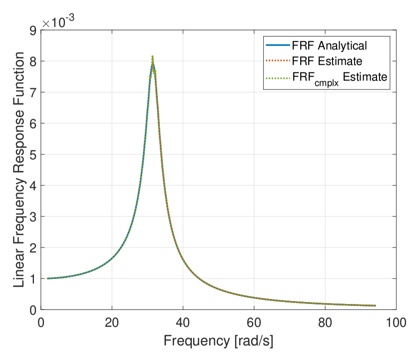
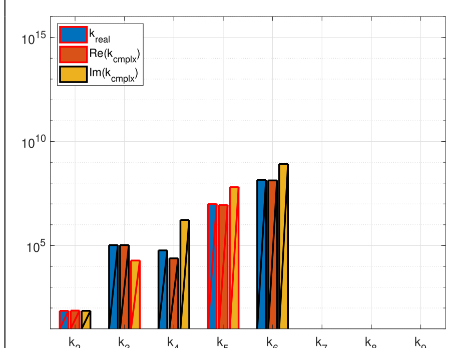
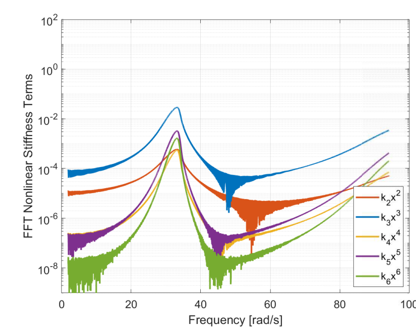
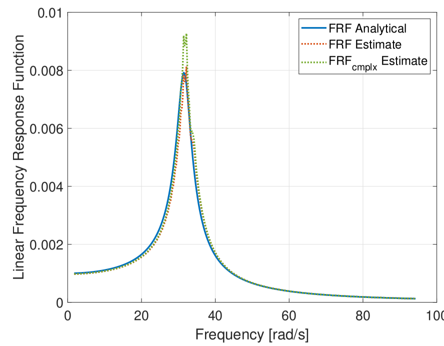
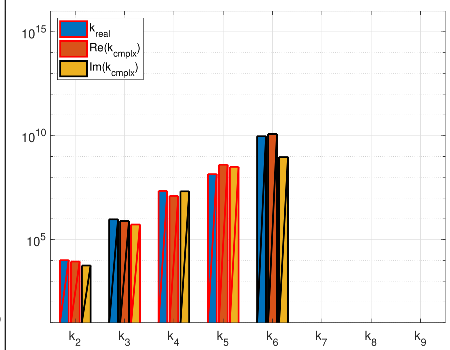
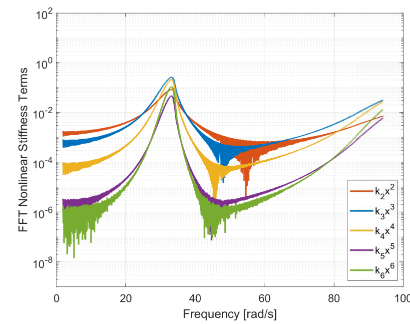
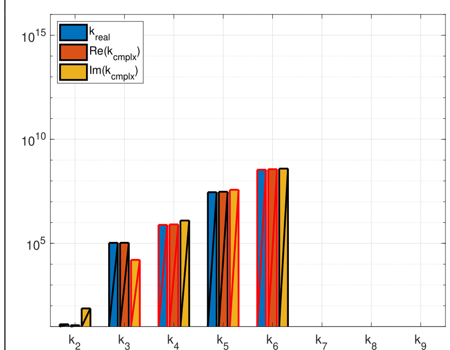
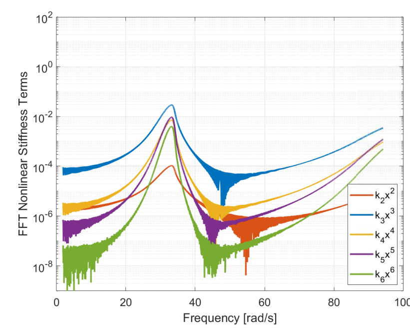
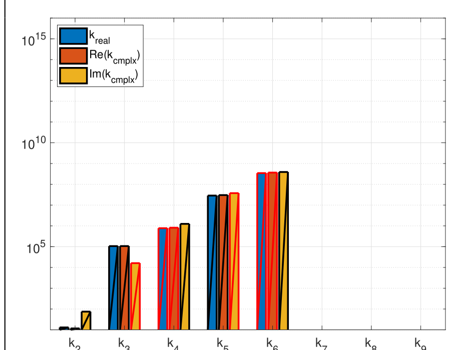
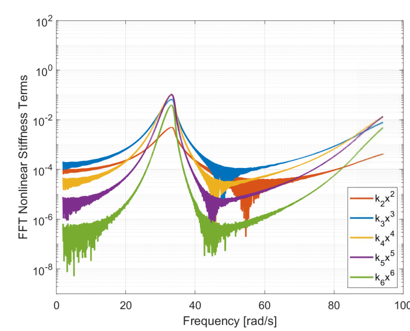
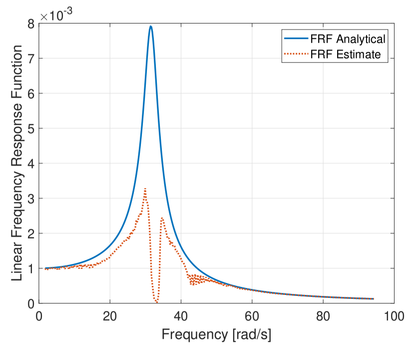
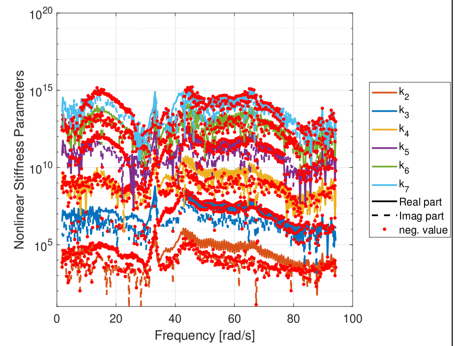
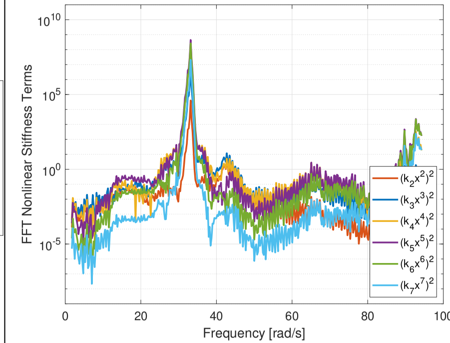
Tab. 8: Degree of stiffness polynomial: 6. Black (red) border around the bars indicates positive (negative) value.

	Linear FRF estimate	Nonlin. param. estimates	Nonlin. terms FFT ($x(t)$ – output signal)
H$_1$-NIFO			
H$_2$-NIFO			
H$_1$-NIXO			

	Linear FRF estimate	Nonlin. param. estimates	Nonlin. terms FFT ($x(t)$ – output signal)
$\mathbf{H}_2$-NIXO			
$\mathbf{H}_1$-NIXO WLDP	×		
$\mathbf{H}_2$-NIXO WLDP	×		

Tab. 9: Degree of stiffness polynomial: 7. Black (red) border around the bars indicates positive (negative) value.

<table>
<tr><th></th><th>Linear FRF estimate</th><th>Nonlin. param. estimates</th><th>Nonlin. terms FFT
($x(t)$ – output signal)</th></tr>
<tr><td>H$\mathbf{H}_1$-NIFO</td><td>
</td><td>
</td><td>
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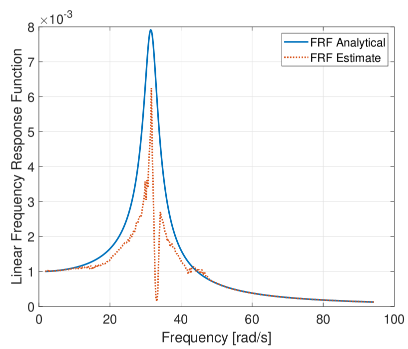
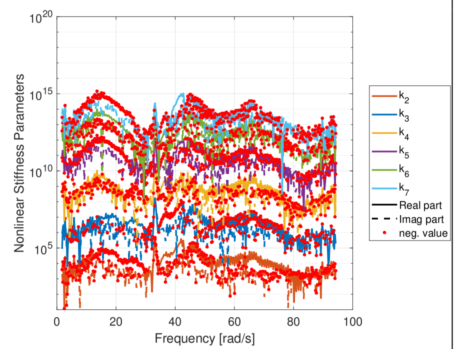
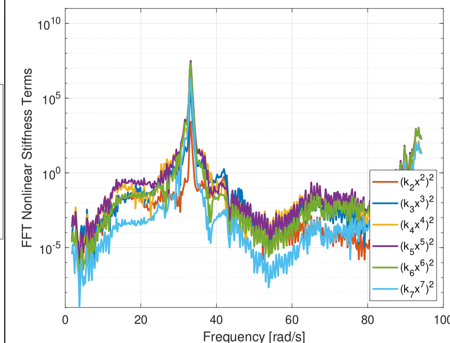
<tr><td>$\mathbf{H}_2$-NIFO</td><td>
</td><td>
</td><td>
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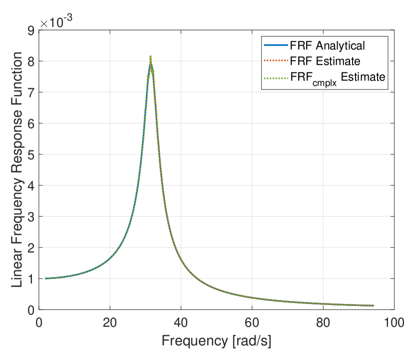
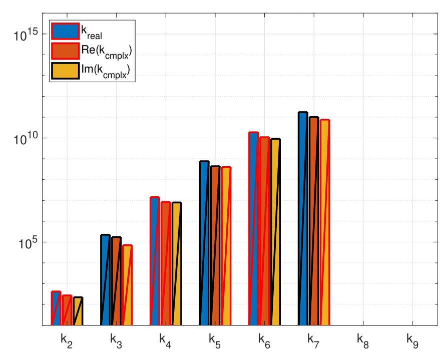
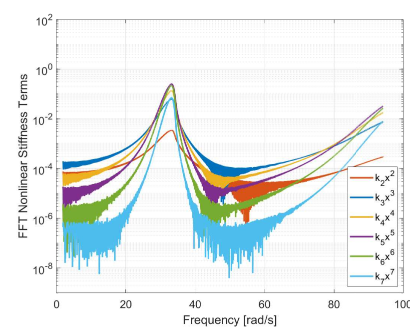
<tr><td>$\mathbf{H}_1$-NIXO</td><td>
</td><td>
</td><td>
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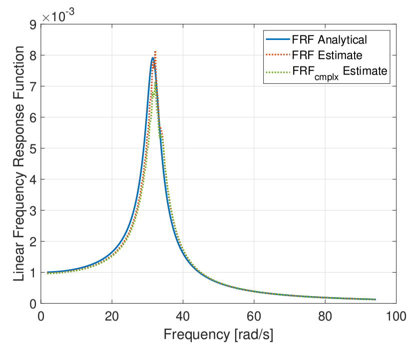
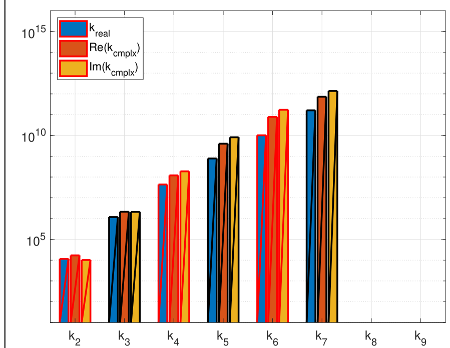
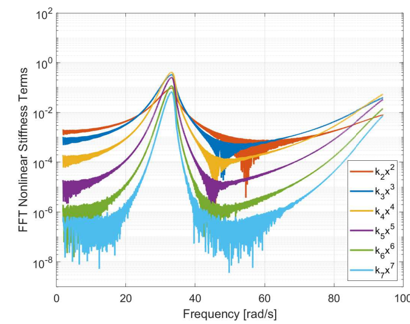
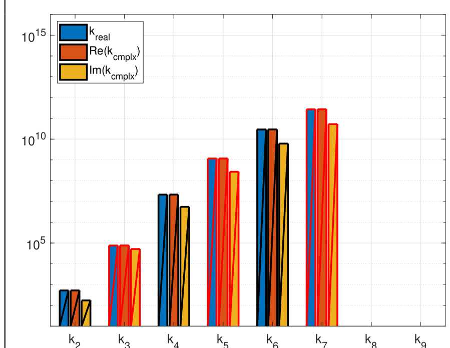
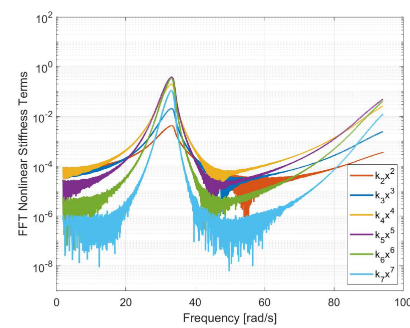
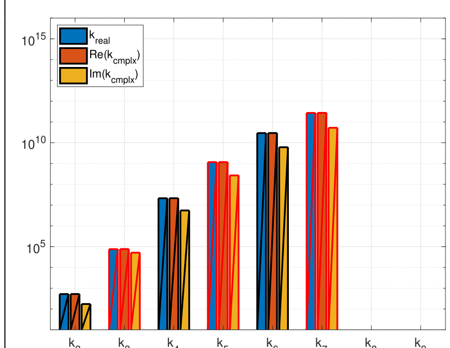
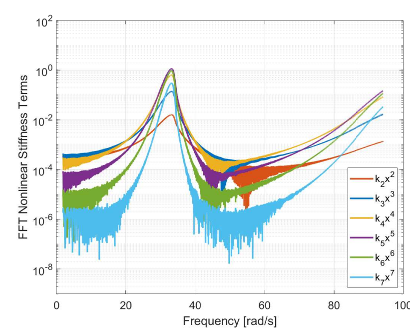
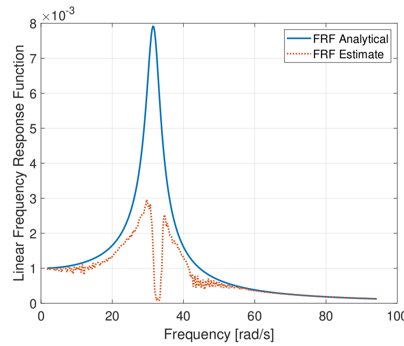
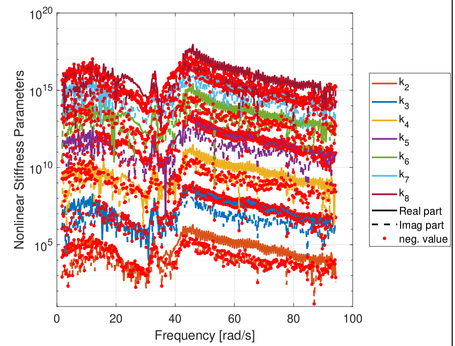
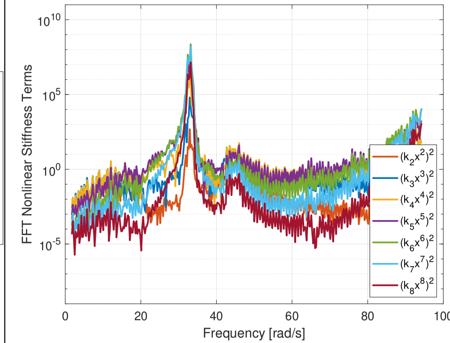
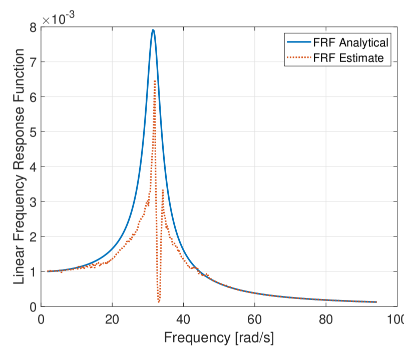
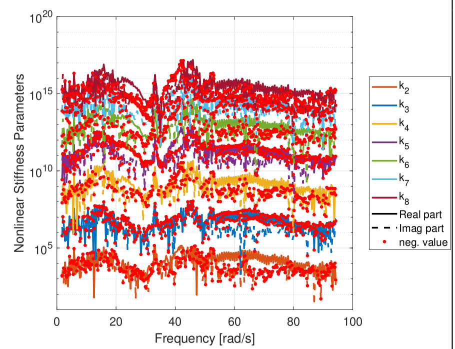
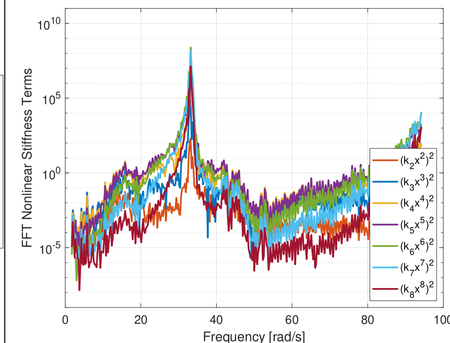
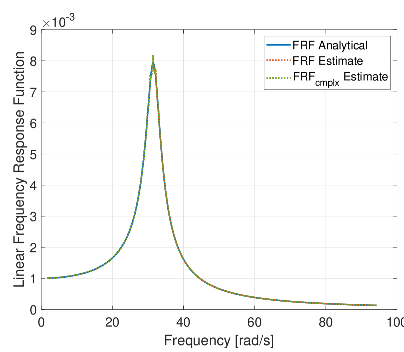
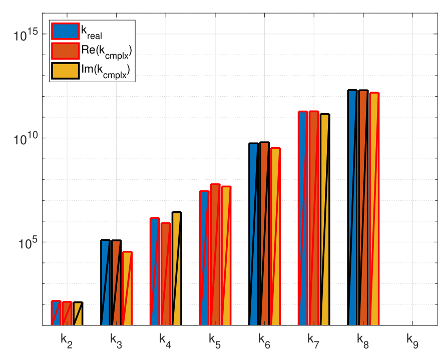
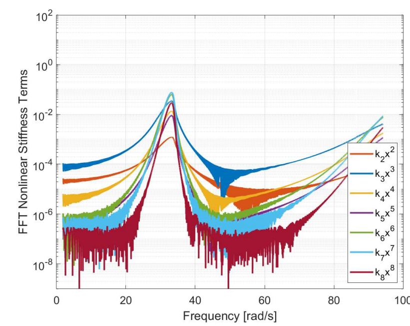
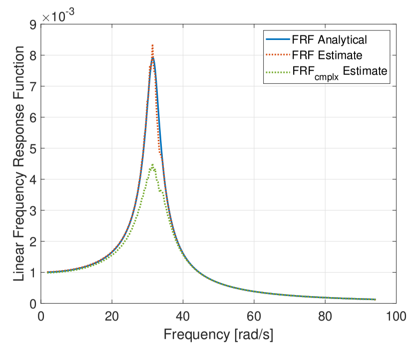
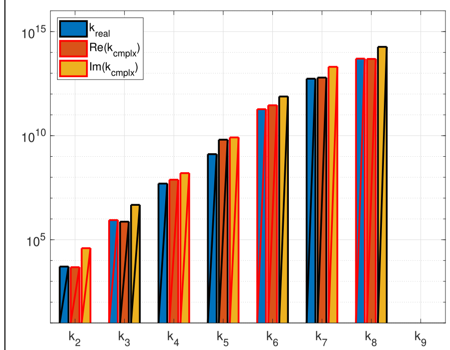
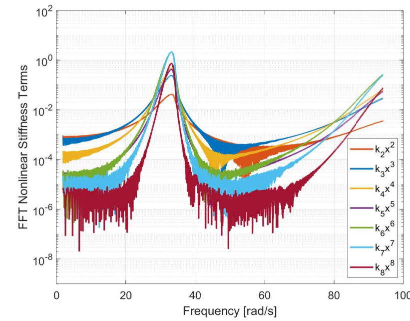
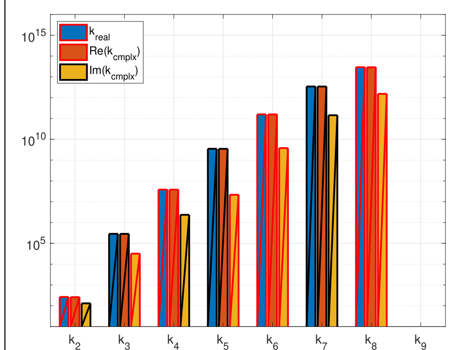
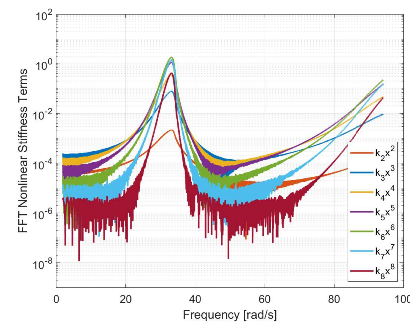
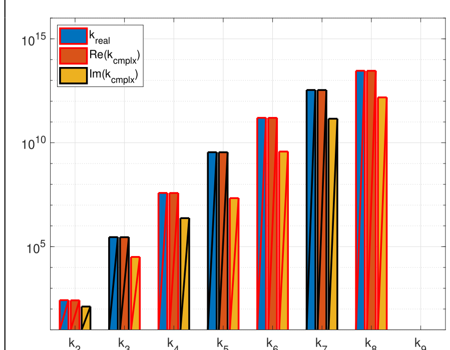
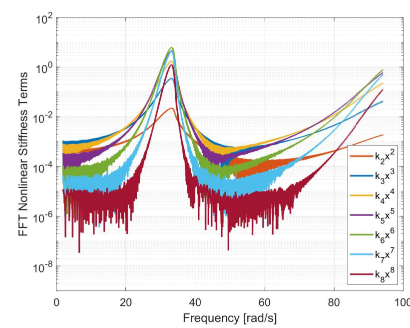
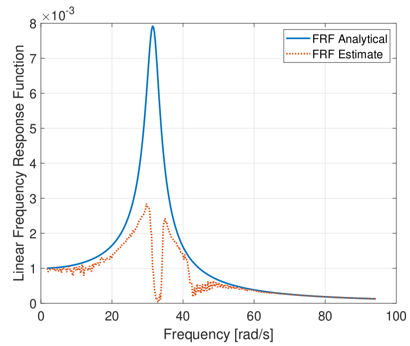
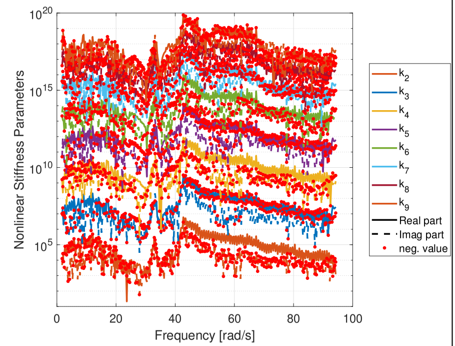
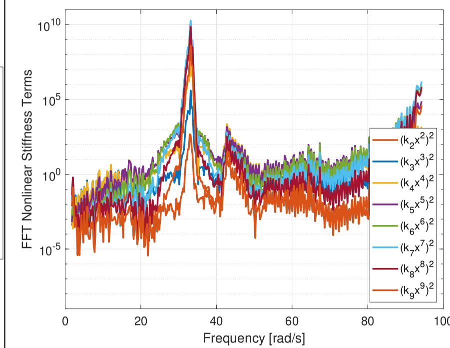
	Linear FRF estimate	Nonlin. param. estimates	Nonlin. terms FFT ($x(t)$ – output signal)
$\mathbf{H}_2$-NIXO			
$\mathbf{H}_1$-NIXO WLDP	×		
$\mathbf{H}_2$-NIXO WLDP	×		

Tab. 10: Degree of stiffness polynomial: 8. Black (red) border around the bars indicates positive (negative) value.

	Linear FRF estimate	Nonlin. param. estimates	Nonlin. terms FFT ($x(t)$ – output signal)
$\mathbf{H}_1$-NIFO			
$\mathbf{H}_2$-NIFO			
$\mathbf{H}_1$-NIXO			

	Linear FRF estimate	Nonlin. param. estimates	Nonlin. terms FFT ($x(t)$ – output signal)
$\mathbf{H}_2$-NIXO			
$\mathbf{H}_1$-NIXO WLDP	×		
$\mathbf{H}_2$-NIXO WLDP	×		

Tab. 11: Degree of stiffness polynomial: 9. Black (red) border around the bars indicates positive (negative) value.

<table>
<tr><th></th><th>Linear FRF estimate</th><th>Nonlin. param. estimates</th><th>Nonlin. terms FFT
($x(t)$ – output signal)</th></tr>
<tr><td>H$\mathbf{H}_1$-NIFO</td><td>
</td><td>
</td><td>
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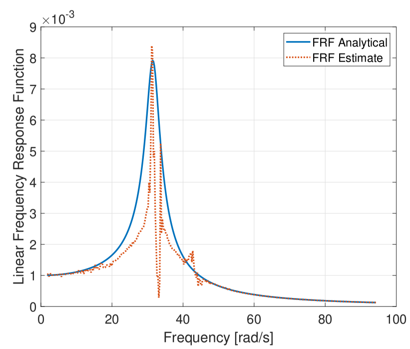
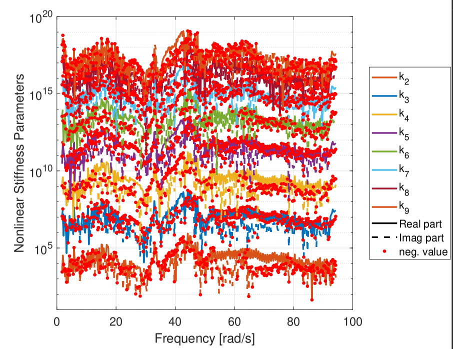
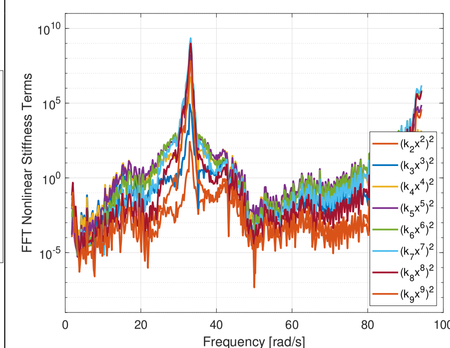
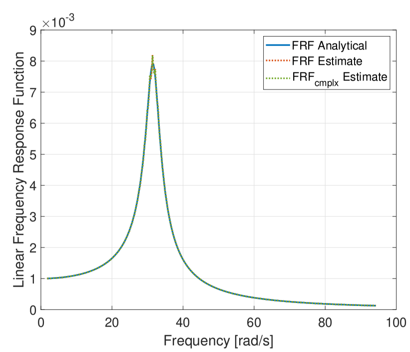
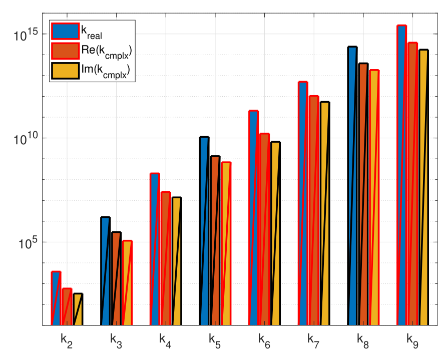
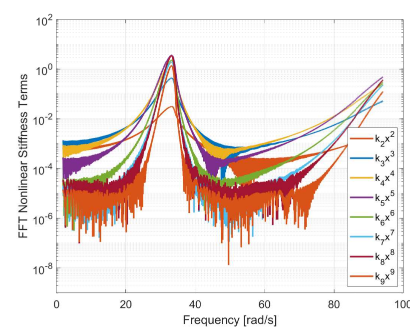
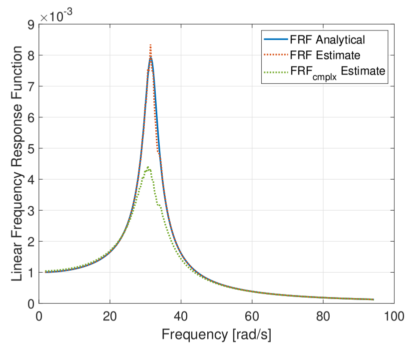
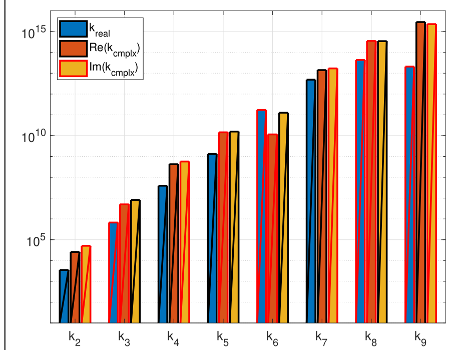
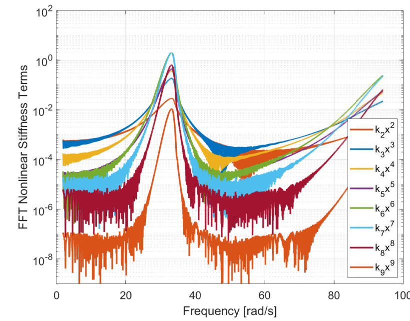
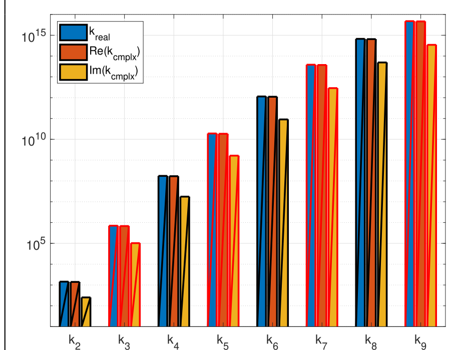
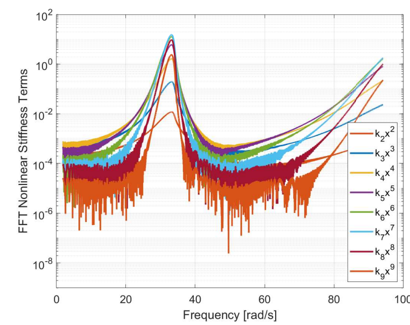
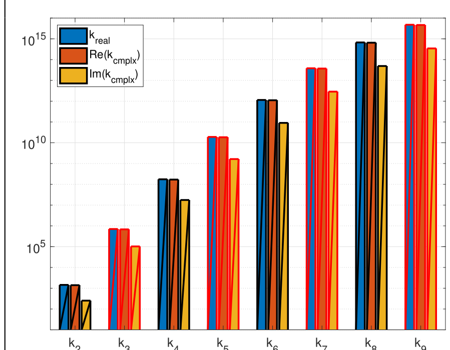
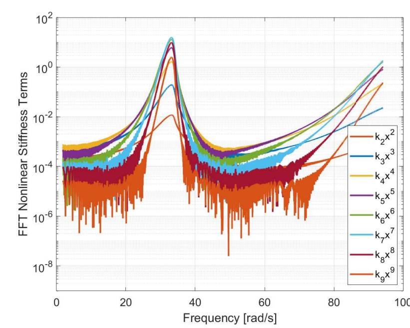
<tr><td>$\mathbf{H}_2$-NIFO</td><td>
</td><td>
</td><td>
</td></tr>
<tr><td>$\mathbf{H}_1$-NIXO</td><td>
</td><td>
</td><td>
</td></tr>
</table>

	Linear FRF estimate	Nonlin. param. estimates	Nonlin. terms FFT ($x(t)$ – output signal)
H$_2$-NIXO			
H$_1$-NIXO WLDP	×		
H$_2$-NIXO WLDP	×		

Comments

1. Results obtained with NIXO methods allow for identification which nonlinear terms are dominant in the mechanical system response. Plots presented in the third column of Tabs. from 5 to 8 clearly show that there is a cubic nonlinearity in the system. These tables show results from analysis where the model function was a polynomial of up to sixth degree. Thus, the nonlinear terms of degree 2, 4, 5 and 6 might be eliminated from the equation of motion, or at least it can be claimed that they have lower impact on the response of the system that the cubic nonlinear term.

2. Comparison of force (or energy) accumulated in the nonlinear springs/dampers might be not the only strategy to eliminate the nonlinear terms which are not describing the mechanical system. A careful reader observed that the result of the NIXO identification process consist of k_{real} and k_{cmplx} parameters. Indeed – NIXO methods allow for estimation of nonlinear parameters as complex numbers, as well as for enforcing them to be found as real numbers.

   The authors of this report believe that if the identification process if conducted on correctly - than the real part of k_{cmplx} should match k_{real}. Moreover, complex part of k_{cmplx} should be significantly smaller than $\mathrm{Re}\{k_{cmplx}\}$. To sum up, to keep the nonlinear term in the mechanical system model, following conditions should be satisfied:

   (a) $k_{real} = \mathrm{Re}\{k_{cmplx}\}$

   (b) $\mathrm{Re}\{k_{cmplx}\} >> \mathrm{Im}\{k_{cmplx}\}$.

3. System identification with polynomials of degree 7, 8 and 9 (see Tabs. from 9 to 11) do not allow to distinguish which nonlinear terms are dominant in the system's response. To investigate if these nonlinear terms should be included in the mechanical system model, one more analysis needs to be conducted on. This time system should be identified with model function (△).

$$m\ddot{x} + c\dot{x} + kx + k_3x^3 + k_r x|x|^{r-1} = f(t), \qquad (\triangle)$$

   where $r \in \{7,\ 8,\ 9\}$. Results from this additional analysis are presented in section 3.

4. Results obtained from system identification using NIFO method do not allow to distinguish which terms are dominant in the mechanical system response. This might be caused by the fact that NIFO methods do not work well when system is excited using swept cosine. They work well if the forcing function is e.g. burst random (see [3] for more details).

3 Case Study – Impact of the Additional Term

Results presented in section 2 showed that the cubic nonlinearity is probably dominant in the mechanical system. This observation is confirmed when the model function used to identify the system is a polynomial of up to sixth degree (see the third column of Tabs. from 5 to 8). When the model function is a polynomial of higher degree (seventh, eighth or ninth), the results do not give any insight which nonlinear terms are clearly dominant in the response and which can be eliminated with confidence.

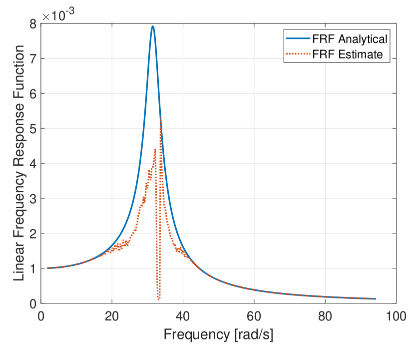
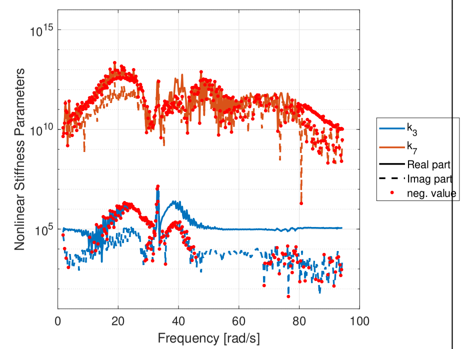
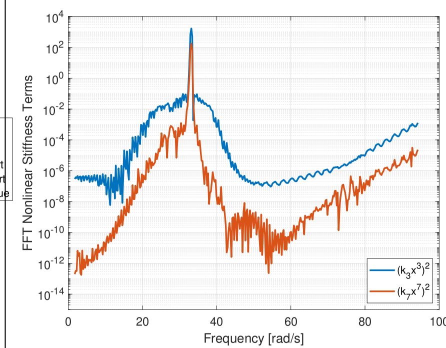
In this section, the identification of mechanical system described in Eq. (1) is continued. Values of its mechnical parameters are preseneted in Tab. 1. This time the model function will contain the cubic term (since it was shown in section 2that the nonlinearity of this degree occurs in the system resonance) and one additional nonlinear term of higher order, as shown in Eq. (4).

$$m\ddot{x} + c\dot{x} + kx + k_3x^3 + k_rx|x|^{r-1} = f(t) \tag{4}$$

Values of powers r are following: $r \in \{7,\ 8,\ 9\}$ and k_3 and k_r's are unknown parameters. Mechanical system is again subjected to swept cosine excitation defied in Eq. (2) and parameters' values summarized in Tab. 2. Time and frequency domain responses of input and output signals are also presented in Tab. 3.

The results are presented in Tabs. from 12 to 14. Results are briefly commented at the end of this section.

Tab. 12: Additional nonlinear term degree: $r = 7$. Black (red) border around the bars indicates positive (negative) value.

<table>
<tr><th></th><th>Linear FRF estimate</th><th>Nonlin. param. estimates</th><th>Nonlin. terms FFT
($x(t)$ – output signal)</th></tr>
<tr><td>H$_1$-NIFO</td><td>
</td><td>
</td><td>
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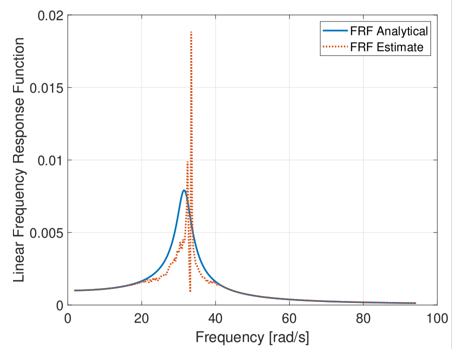
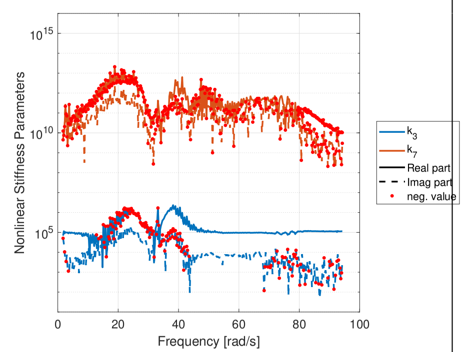
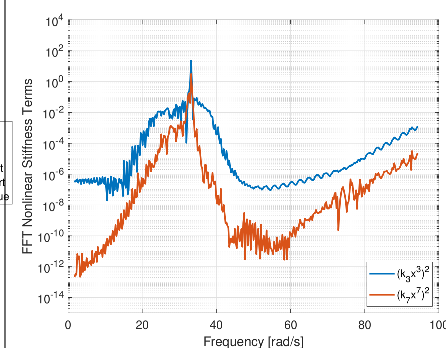
<tr><td>H$_2$-NIFO</td><td>
</td><td>
</td><td>
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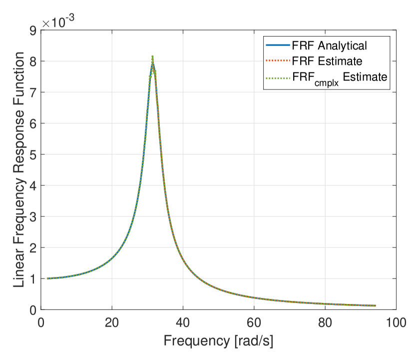
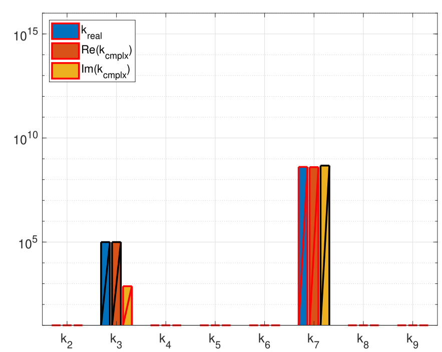
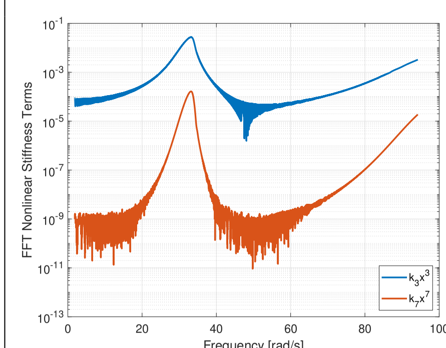
<tr><td>H$_1$-NIXO</td><td>
</td><td>
</td><td>
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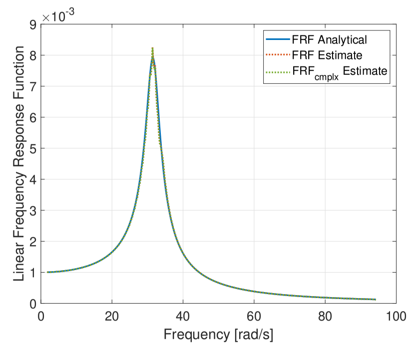
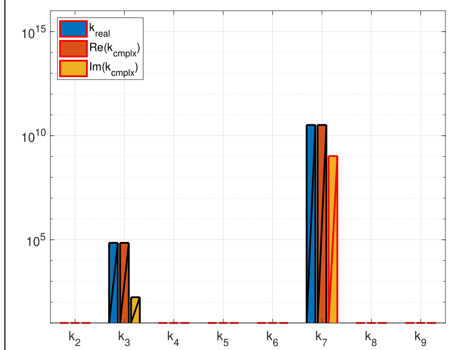
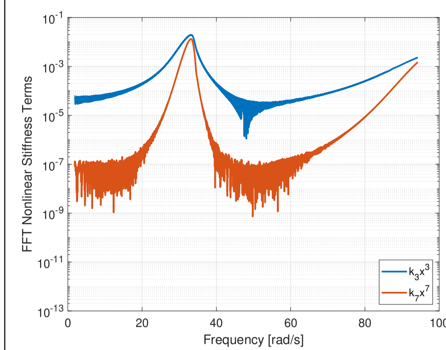
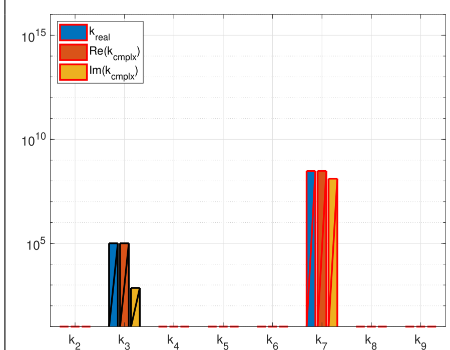
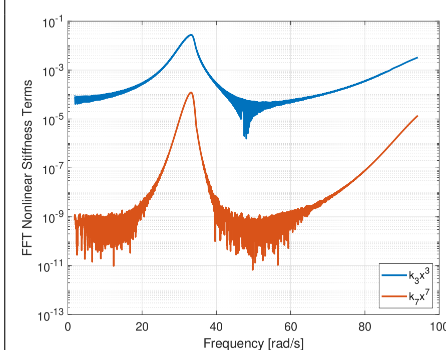
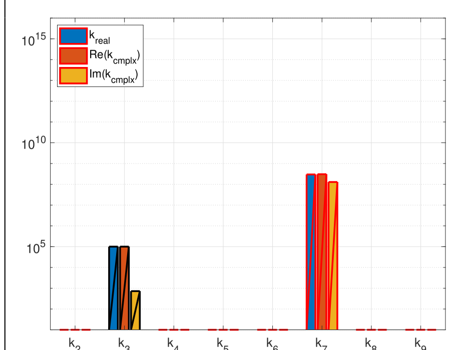
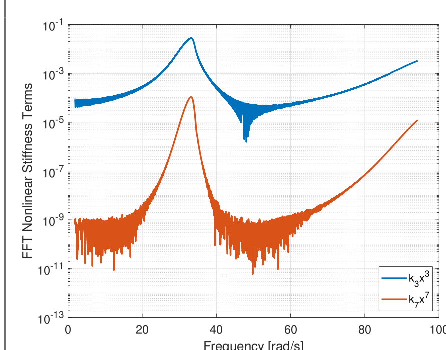
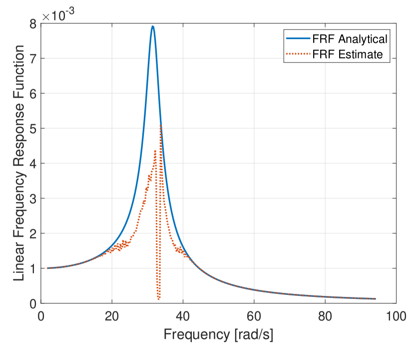
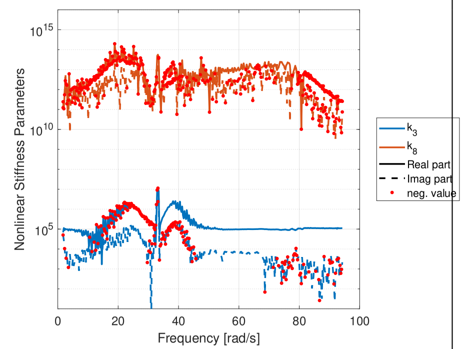
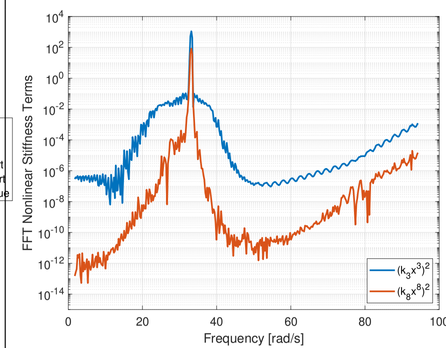
</table>

	Linear FRF estimate	Nonlin. param. estimates	Nonlin. terms FFT ($x(t)$ – output signal)
$\mathbf{H}_2$-NIXO			
$\mathbf{H}_1$-NIXO WLDP	×		
$\mathbf{H}_2$-NIXO WLDP	×		

Tab. 13: Additional nonlinear term degree: $r = 8$. Black (red) border around the bars indicates positive (negative) value.

<table>
<tr><th></th><th>Linear FRF estimate</th><th>Nonlin. param. estimates</th><th>Nonlin. terms FFT
($x(t)$ – output signal)</th></tr>
<tr><td>H$\mathbf{H}_1$-NIFO</td><td>
</td><td>
</td><td>
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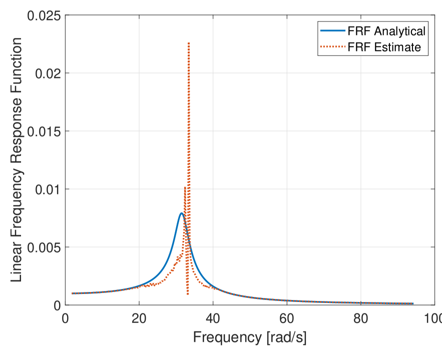
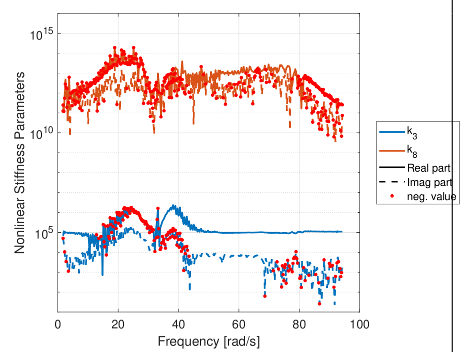
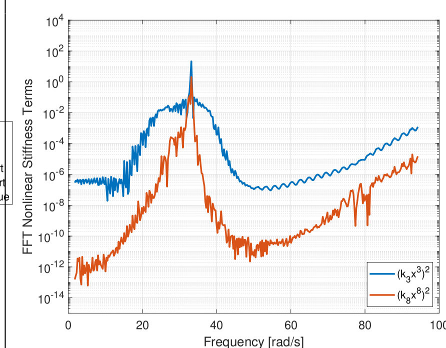
<tr><td>$\mathbf{H}_2$-NIFO</td><td>
</td><td>
</td><td>
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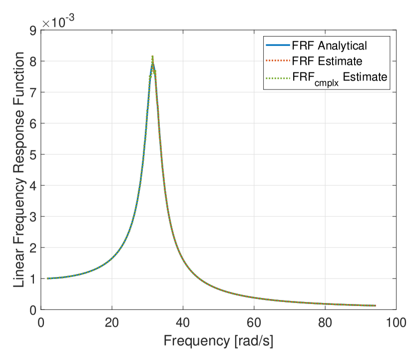
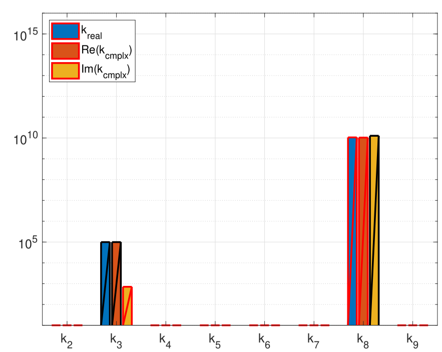
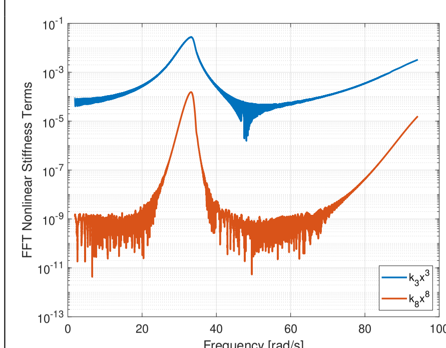
<tr><td>$\mathbf{H}_1$-NIXO</td><td>
</td><td>
</td><td>
</td></tr>
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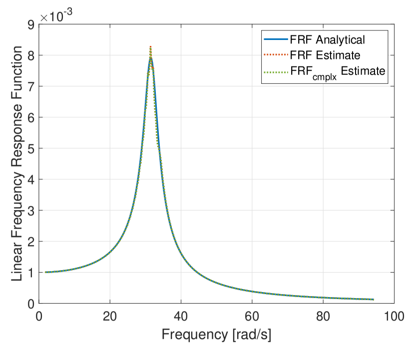
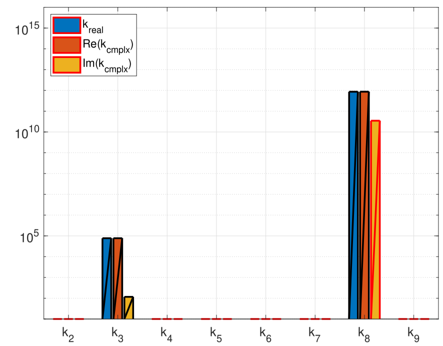
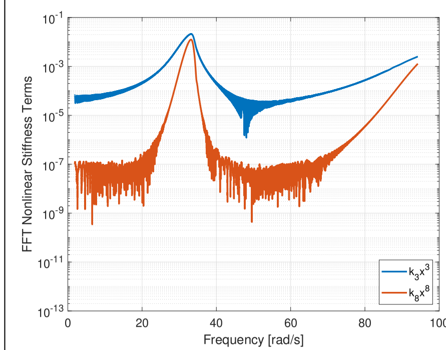
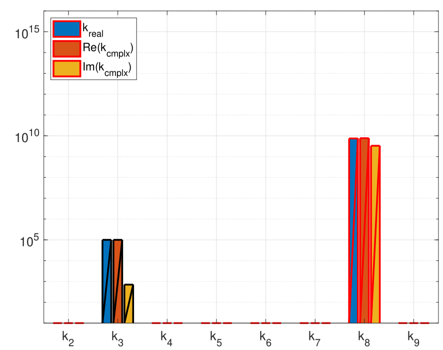
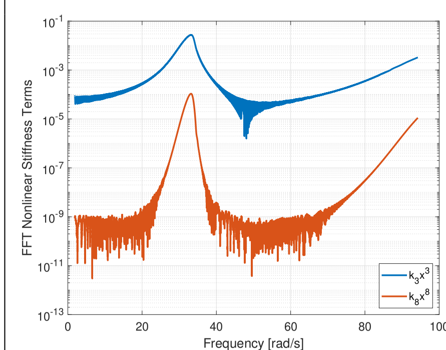
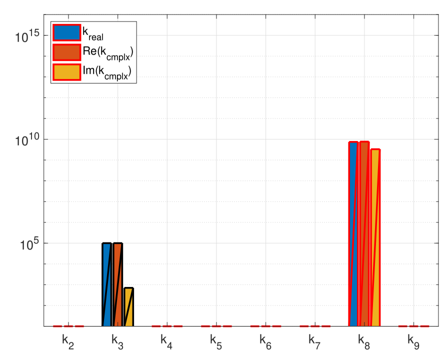
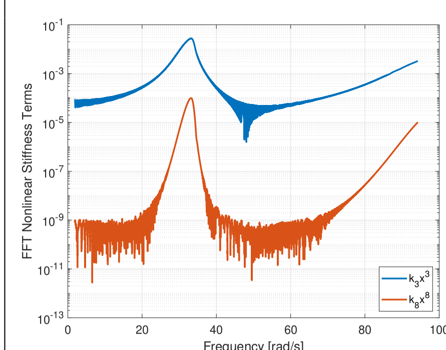
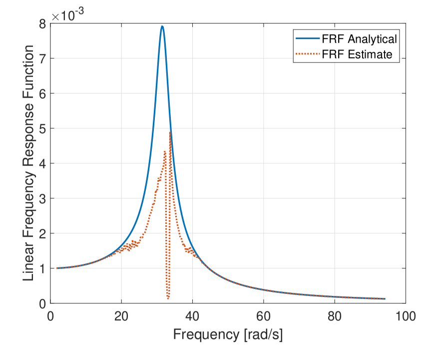
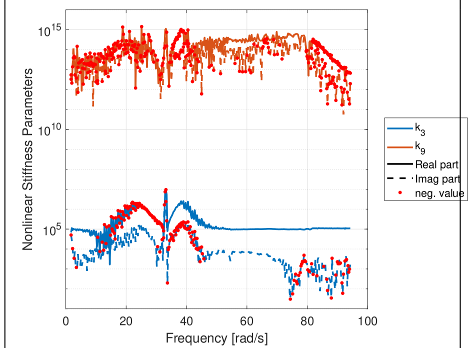
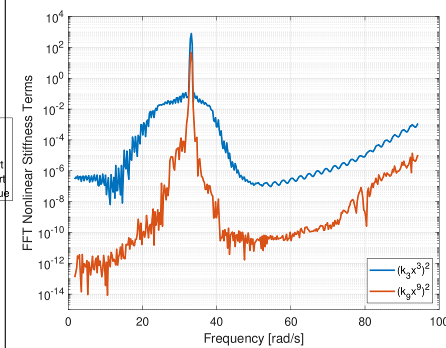
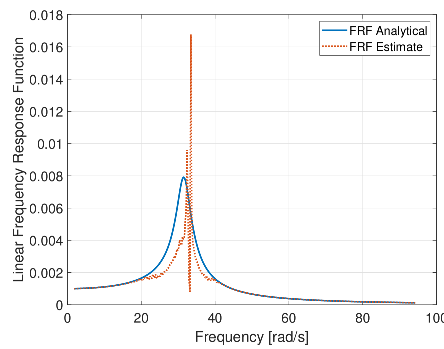
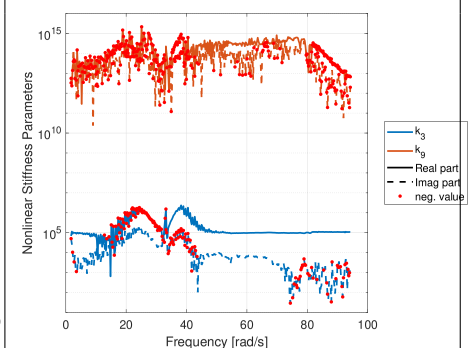
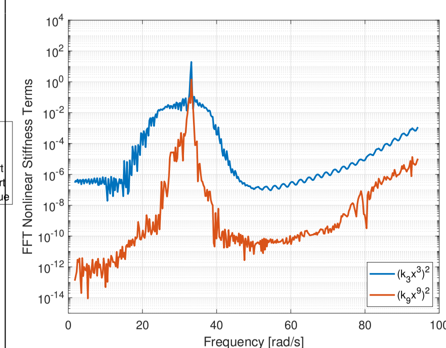
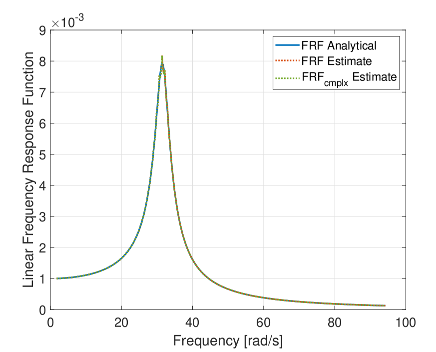
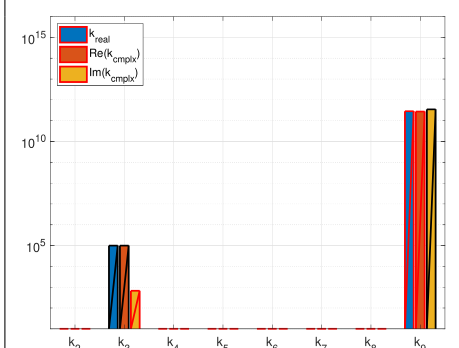
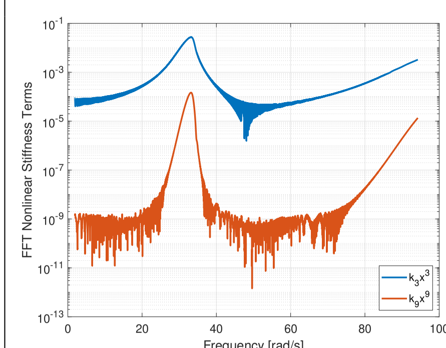
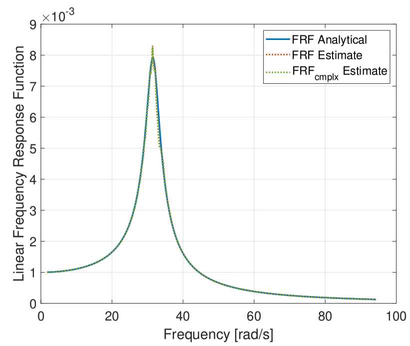
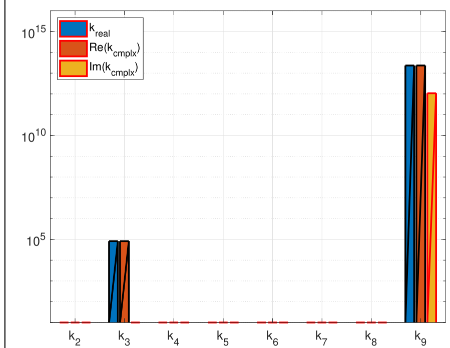
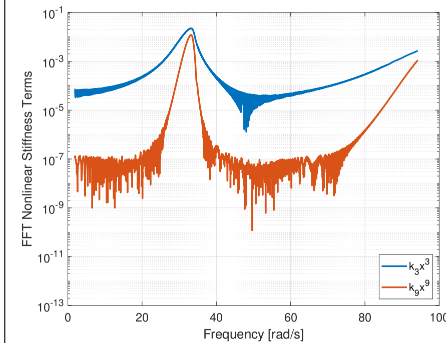
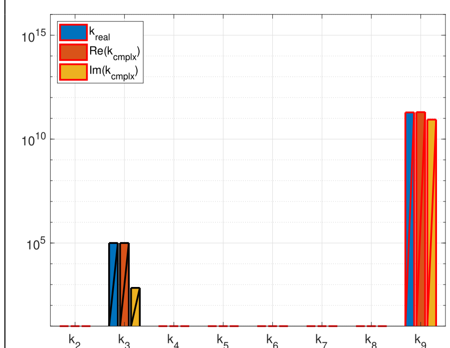
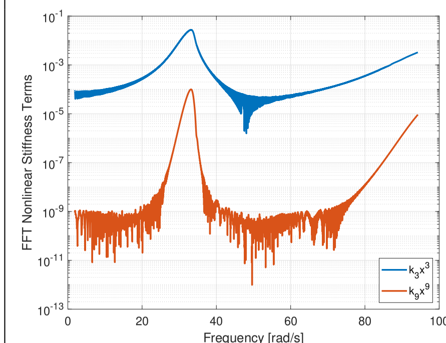
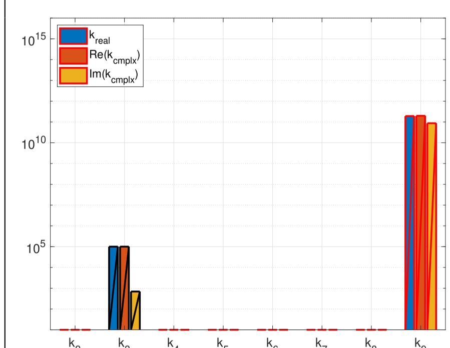
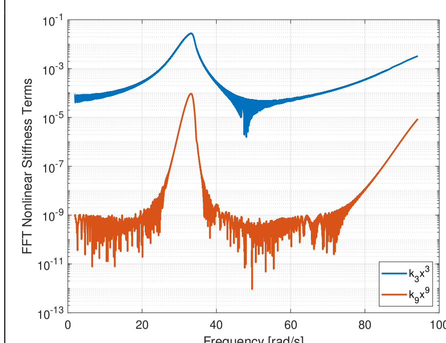
	Linear FRF estimate	Nonlin. param. estimates	Nonlin. terms FFT ($x(t)$ – output signal)
H_2-NIXO			
H_1-NIXO WLDP	×		
H_2-NIXO WLDP	×		

Tab. 14: Additional nonlinear term degree: $r = 9$. Black (red) border around the bars indicates positive (negative) value.

	Linear FRF estimate	Nonlin. param. estimates	Nonlin. terms FFT ($x(t)$ – output signal)
$\mathbf{H}_1$-NIFO			
$\mathbf{H}_2$-NIFO			
$\mathbf{H}_1$-NIXO			

	Linear FRF estimate	Nonlin. param. estimates	Nonlin. terms FFT ($x(t)$ – output signal)
H_2-NIXO			
H_1-NIXO WLDP	×		
H_2-NIXO WLDP	×		

Comments

1. NIFO and NIXO methods find values of k_3 with a satisfactory precision (note that NIFO methods give an accurate estimates for the off-resonant frequency range).
2. Cubic nonlinear term is more dominant than the terms of degree 7, 8 and 9. Force (or energy) accumulated in the cubic spring is significantly higher than the one accumulated in the other springs.
3. Moreover, in some cases the imaginary part of $k_{r,cmplx}$ ($r \in \{7, 8, 9\}$) is not significantly smaller than the $\mathrm{Re}\{k_{r,cmplx}\}$. This might also show that nonlinear terms of higher order are not contained in the mechanical system. Observe that in all the investigated cases, $\mathrm{Im}\{k_{r,cmplx}\}$ is at least 100 times lower than $\mathrm{Re}\{k_{r,cmplx}\}$.

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