Experimental Modal Substructuring with Nonlinear Modal Iwan Models to Capture Nonlinear Subcomponent Damping

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ABSTRACT

This work proposes a means whereby weak nonlinearity in a substructure, as typically arises due to friction in bolted interfaces, can be captured experimentally on a mode-by-mode basis and then used to predict the nonlinear response of an assembly. The method relies on the fact that the modes of a weakly nonlinear structure tend to remain uncoupled so long as their natural frequencies are distinct and higher harmonics generated by the nonlinearity do not produce significant response in other modes. Recent experiments on industrial hardware with bolted joints has shown that this type of model can be quite effective, and that a single degree-of-freedom (DOF) system with an Iwan joint, which is known as a modal Iwan model, effectively captures the way in which the stiffness and damping depend on amplitude. Once the modal Iwan models have been identified for each mode of the subcomponent(s) of interest, they can be assembled using standard techniques and used with a numerical integration routine to compute the nonlinear transient response of the assembled structure. The proposed methods are demonstrated by coupling a modal model of a 3DOF system with three discrete Iwan joints to a linear model for a 2DOF system.

Keywords: reduced order modeling, friction, interface, nonlinear modes, complexification and averaging.

1. Introduction

Experimental-analytical substructuring allows one to couple an experimentally derived model for a structure that is difficult to model, with a finite element model for the rest of the assembly in order to predict the system's response. While there are countless compelling industrial applications, many of the systems that are most difficult to model, and hence where experimental-analytical substructuring would be most beneficial, contain many interfaces with bolted joints. Interfaces in built up structures are responsible for much of the damping in the assembly, and are the most common source of nonlinearity. This work presents an extension of modal substructuring for this class of structure.

Recent works have shown that bolted interfaces can cause the damping in a system to increase by a factor of two or more (see, e.g. [1-3]), while the effective natural frequency tends to change relatively little. Furthermore, under the conditions outlined in [4] (simplistically that the joint forces and their harmonics are distinct from each modal frequency), the modes of the structure tend to remain uncoupled, so that the structure can be modeled accurately using a collection of uncoupled, weakly-nonlinear oscillators [5, 6]. This was thoroughly confirmed in [1] for an assembly of automotive exhaust components, by exciting the structure at multiple locations and various force levels (in the micro-slip regime). A second investigation on a cylindrical structure with bolted joints and nonlinear contact between foam and an internal structure also highlighted the usefulness of this approach [7].

In this work we propose to use this class of model (e.g. uncoupled SDOF oscillators) to represent a subcomponent and then to assemble that subcomponent to the rest of the structure of interest. Specifically, the set of nonlinear oscillators are assembled using standard finite element assembly techniques. The assembled equation of motion and its Jacobian are then used in a Newmark integration routine to predict the transient response of the assembled structure. The methods are tested through simulations on a simple spring mass system. While the method is applicable for a wide range of nonlinear SDOF oscillator models, this work uses a modal Iwan model for each subcomponent. This type of model accurately captures the power-law dependence of damping on amplitude that is frequently observed in experiments [1, 5, 8, 9]. Other SDOF models, some of which may be simpler or less expensive to use, were evaluated in a recent study by the authors [7].

The paper is organized as follows. Section 2 outlines the approach used. In Section 3 the proposed techniques are validated by deriving modal Iwan models for the three modes of a 3DOF system, which is then assembled to a linear 2DOF system. The conclusions are then presented in Section 4.

2. Theoretical Development

In the most general case, the equation of motion for substructure *A* can be written as follows,

$$\mathbf{M}^A\ddot{\mathbf{x}}^A + \mathbf{C}^A\dot{\mathbf{x}}^A + \mathbf{K}^A\mathbf{x}^A + \sum_{k=1}^{N_J} \mathbf{f}_{J,k}^A f_{J,k}^A\left(\mathbf{x}^A, \phi_{k,1}^A \ldots \phi_{k,N_j}^A\right) = \mathbf{f}^A(t) \tag{1}$$

where $\mathbf{M}^A$, $\mathbf{C}^A$ and $\mathbf{K}^A$ are the $N \times N$ linear mass, damping and stiffness matrices and the *k*th scalar joint force, $f_{J,k}^A\left(\mathbf{x}^A, \phi_{k,1}^A \ldots \phi_{k,N_j}^A\right)$ depends on the displacement vector $\mathbf{x}^A$ and on its internal slider states $\phi_{k,1}^A \ldots \phi_{k,N_j}^A$. The constant vector $\mathbf{f}_{J,k}^A$ maps each scalar joint force $f_{J,k}^A$ to the points to which the joint is attached. For example, in the example that will be discussed later, shown in Fig. 1 the first Iwan joint is between DOF 1 and ground so $\mathbf{f}_{J,1}^A = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^{\mathrm{T}}$ and the third Iwan joint is connected between DOF 2 and 3, so $\mathbf{f}_{J,3}^A = \begin{bmatrix}0 & 1 & -1\end{bmatrix}^{\mathrm{T}}$. Similar equations could be written for substructures *B*, *C*, etc….

When each mode of the substructure is represented as a modal Iwan model, the matrices **M**, **C** and **K** would be diagonal and the *k*th joint force would depend on only one modal displacement.

We shall employ a primal formulation [10] to couple the substructures. Without loss of generality, consider the case where substructure *A* will be joined to substructure *B*. The substructures can be coupled by writing constraint equations of the following form,

$$\mathbf{B}\begin{bmatrix}\mathbf{x}^A \\ \mathbf{x}^B\end{bmatrix} = 0 \tag{2}$$

and then eliminating the redundant degrees of freedom using

$$\begin{bmatrix}\mathbf{x}^A \\ \mathbf{x}^B\end{bmatrix} = \mathbf{L}\mathbf{q} \tag{3}$$

$$\mathbf{L} = null\left(\mathbf{B}\right) \tag{4}$$

to obtain a set of independent (or unconstrained [11]) coordinates, **q**. The equations of motion for the coupled system then become the following in terms of the coordinates **q**,

$$\hat{\mathbf{M}}\ddot{\mathbf{q}} + \hat{\mathbf{C}}\dot{\mathbf{q}} + \hat{\mathbf{K}}\mathbf{q} + \mathbf{L}^{\mathrm{T}}\begin{bmatrix}\sum_{k=1}^{N_J} \mathbf{f}_{J,k}^A f_{J,k}^A\left(\mathbf{x}^A, \phi_{k,1}^A \ldots \phi_{k,N_j}^A\right) \\ \sum_{k=1}^{N_J} \mathbf{f}_{J,k}^B f_{J,k}^B\left(\mathbf{x}^B, \phi_{k,1}^B \ldots \phi_{k,N_j}^B\right)\end{bmatrix} = \mathbf{L}^{\mathrm{T}}\begin{bmatrix}\mathbf{f}^A(t) \\ \mathbf{f}^B(t)\end{bmatrix} \tag{5}$$

where

$$\hat{\mathbf{M}} = \mathbf{L}^{\mathrm{T}} \begin{bmatrix} \mathbf{M}^A & 0 \\ 0 & \mathbf{M}^B \end{bmatrix} \mathbf{L} \tag{6}$$

and similarly for $\hat{\mathbf{C}}$ and $\hat{\mathbf{K}}$. Further details can be found in [10] or ([11], Chapter 9).

In order to simulate the response of the assembly, the unconditionally stable Newmark algorithm [12] is used (e.g. with $\beta_N = 0.25$ and $\gamma = 0.5$). This procedure was first developed by Simmermacher as reported in [8]. A Newton iteration loop is used to adjust the displacement of the joint (and the internal slider states) so that the joint force is in dynamic equilibrium at each time step. Specifically, if the displacement at the jth time step is denoted $\mathbf{q}_j$, then the residual is defined as.

$$\mathbf{r}_j = \hat{\mathbf{M}}\ddot{\mathbf{q}}_j + \hat{\mathbf{C}}\dot{\mathbf{q}}_j + \hat{\mathbf{K}}\mathbf{q}_j + \mathbf{L}^{\mathrm{T}} \begin{bmatrix} \sum_{k=1}^{N_J} \mathbf{f}_{J,k}^A f_{J,k}^A \left(\mathbf{L}\mathbf{q}_j, \phi_{k,1}^A \ldots \phi_{k,N_j}^A \right) \\ \sum_{k=1}^{N_J} \mathbf{f}_{J,k}^B f_{J,k}^B \left(\mathbf{L}\mathbf{q}_j, \phi_{k,1}^B \ldots \phi_{k,N_j}^B \right) \end{bmatrix} - \mathbf{L}^{\mathrm{T}} \begin{bmatrix} \mathbf{f}^A(t) \\ \mathbf{f}^B(t) \end{bmatrix} \tag{7}$$

Then, the Jacobian is

$$\mathbf{J}_j = \hat{\mathbf{M}} + \gamma \Delta t \hat{\mathbf{C}} + \beta (\Delta t)^2 \left(\hat{\mathbf{K}} + \mathbf{L}^{\mathrm{T}} \begin{bmatrix} \sum_{k=1}^{N_J} \mathbf{f}_{J,k}^A K_{J,k}^A \\ \sum_{k=1}^{N_J} \mathbf{f}_{J,k}^B K_{J,k}^B \end{bmatrix} \begin{bmatrix} \sum_{k=1}^{N_J} \mathbf{f}_{J,k}^A K_{J,k}^A \\ \sum_{k=1}^{N_J} \mathbf{f}_{J,k}^B K_{J,k}^B \end{bmatrix}^{\mathrm{T}} \mathbf{L} \right) \tag{8}$$

where $K_{J,k}^A$ is the instantaneous stiffness of the kth Iwan joint and depends on the corresponding slider states, $\phi_{k,1}^A \ldots \phi_{k,N_j}^A$. The estimate of the acceleration, displacement and velocity at this time step are updated as follows. For the first iteration the same procedure is used, only with $\mathbf{r}_j = 0$.

$$\begin{aligned} \ddot{\mathbf{q}}_{j,new} &= \ddot{\mathbf{q}}_j - \mathbf{J}_j \mathbf{r}_j \\ \dot{\mathbf{q}}_{j,new} &= \dot{\mathbf{q}}_j + \Delta t (1-\gamma) \ddot{\mathbf{q}}_{j-1} + \gamma \ddot{\mathbf{q}}_{j,new} \\ \mathbf{q}_{j,new} &= \mathbf{q}_j + \Delta t \dot{\mathbf{q}}_{j-1} + \tfrac{1}{2} (\Delta t)^2 (1 - 2\beta_N) \ddot{\mathbf{q}}_{j-1} + 2\beta_N \ddot{\mathbf{q}}_{j,new} \end{aligned} \tag{9}$$

Note that $\mathbf{L}^{\mathrm{T}} \begin{bmatrix} \mathbf{f}_{J,k}^A \\ \mathbf{0} \end{bmatrix}$ and $\mathbf{L}^{\mathrm{T}} \begin{bmatrix} \mathbf{f}_{J,k}^A \left(\mathbf{f}_{J,k}^A \right)^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{L}$, and similarly for substructure B, are simply constant matrices that map each joint force onto the appropriate degrees of freedom in the assembled system. These matrices, and the assembled system matrices $\hat{\mathbf{M}}$, $\hat{\mathbf{C}}$ and $\hat{\mathbf{K}}$ are calculated in advance and only the joint forces and stiffnesses need to be updated in each iteration.

2.1 Iwan Joint

The preceding discussion is valid for a variety of joint models. In this work the Iwan model is used, so each joint can be characterized by four parameters F_s, K_T, χ and β [13]. The first two parameters describe, respectively, the force at which the joint slips completely (macro-slip) and the stiffness of the joint when all sliders are stuck. The model exhibits energy dissipation per cycle, D, that depends on magnitude of the displacement $|x|$ in a power-law fashion as

$$D = R|x|^{3+\chi} \tag{10}$$

where R is a constant. By analogy with a linear system, the effective damping ratio ζ of an SDOF system with mass m and with an Iwan joint in parallel with a spring of stiffness K_0 is the following,

$$\zeta = D / \left(m 2\pi \omega_d \omega_n \left| x \right|^2 \right) \tag{11}$$

where $\omega_n{}^2 = \left(K_0 + K_T \right) / m$ and $\omega_d = \omega_n \sqrt{1-\zeta^2}$. These relationships together with a Hilbert transform were used to fit an Iwan model to simulated measurements of each substructure. For further details, see [1, 5].

3. Simulated Application

The proposed approach was applied to the system depicted in Figure 1.

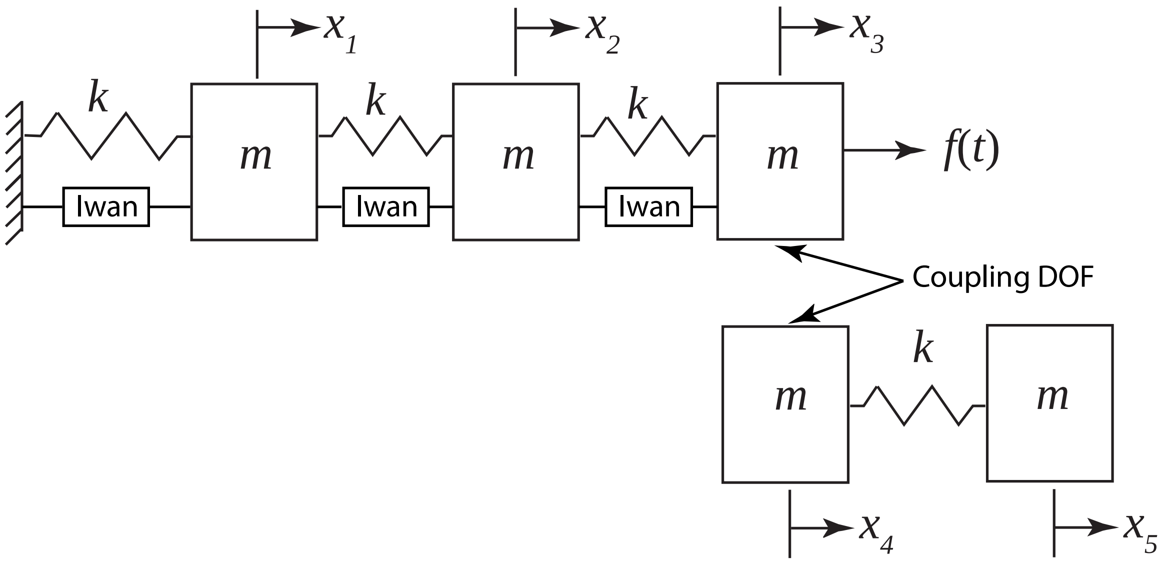

Figure 1. Schematic of the discrete system used to validate the proposed substructuring procedure. (top-left) Substructure *A*, (bottom-right) Substructure *B*. Mass and stiffness proportional damping was added to simulate material damping (dashpots not shown).

Substructure *A* consists of three masses connected by linear springs of stiffness k in parallel with Iwan elements with the parameters shown in Table 1. The other system parameters are m=10 kg, k=5 N/m, $\boldsymbol{C}^A$=0.002($\mathbf{M}^A$+$\mathbf{K}^A$), $\boldsymbol{C}^B$=0.002$\mathbf{K}^B$. The goal is to simulate a test on Substructure A to determine modal Iwan models for each mode of that substructure, and then to utilize modal substructuring to predict the response of the assembly when the masses are joined as indicated with x_3=x_4.

Table 1. Parameters of Iwan Joints in Substructure A.

Iwan Joint	F_S	K_T	χ	β
$\boldsymbol{x_1}$ **– ground**	10 N	5 N/m	-0.5	0.1
$\boldsymbol{x_1 - x_2}$	1 N	4 N/m	-0.2	0.01
$\boldsymbol{x_2 - x_3}$	100 N	3 N/m	-0.8	1

3.1 Estimating Modal Iwan Models for Substructure A

The linear mode shapes $\left[\boldsymbol{\varphi}_1 \;\; \boldsymbol{\varphi}_2 \;\; \boldsymbol{\varphi}_3 \right]$ of Substructure A were assumed to be known (e.g. having been measured from a low-amplitude linear test). Note that in such a test each Iwan joint acts as linear spring with stiffness K_T. Then, to identify a nonlinear model for Substructure A, an experiment was simulated in which a half-sine impulse with a 0.1 second long period and amplitude of 100N was applied to mass 3. The Newmark routine was used to determine the transient response and then the response of each mode was estimated using $\mathbf{q} = \boldsymbol{\varphi}^{-1}\mathbf{x}$. Note that the mode matrix used in this calculation corresponds to the linear, low amplitude modes that include the stiffness of the joints. The FFT $Q_r(\omega)$=FFT(q_r(t)) of each modal response is shown in Figure 2. A weak nonlinearity, as is typical of a structure with bolted joints, is visible near each peak.

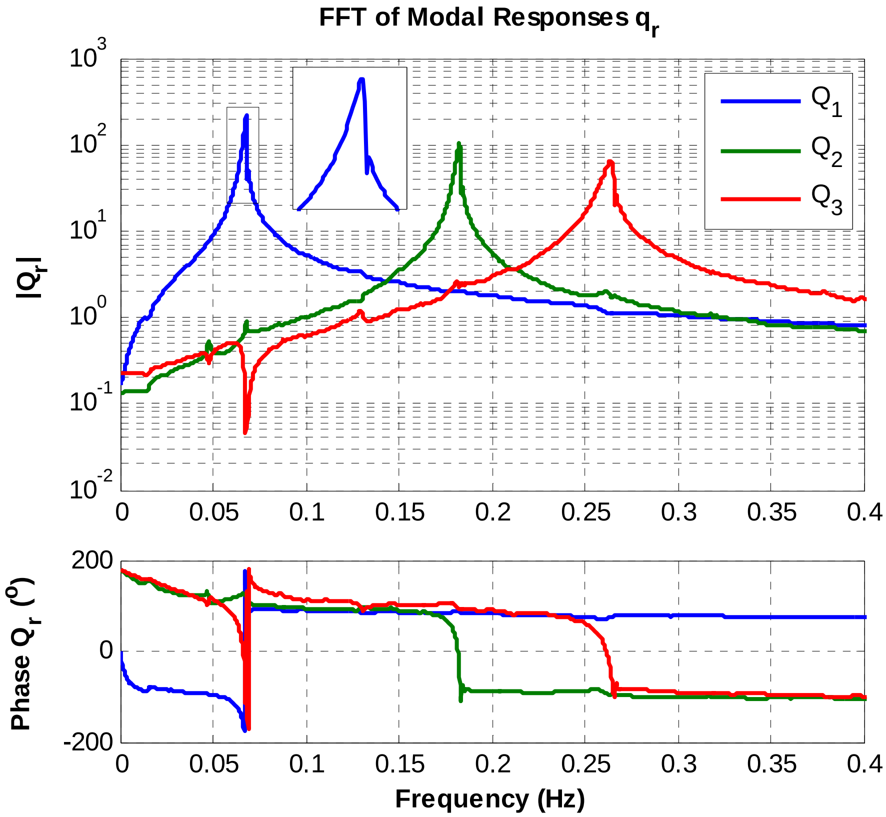

Figure 2. Fast Fourier Transform of the modal response of substructure A.

The simulated measurements were then post processed using the procedure outlined in [1] to identify modal Iwan parameters for each mode. Briefly, each mode's response was band-pass filtered and a smoothed Hilbert transform was used to estimate the instantaneous phase and amplitude as a function of time. The derivative of the phase gives the damped natural frequency, $\omega_{\mathrm{d}} \approx \omega_{\mathrm{n}}$, as a function of time, and the derivative of the amplitude gives $\zeta(t)\omega_{\mathrm{n}}(t)$, from which the damping can be determined. Then the frequency and damping were plotted versus amplitude to determine the modal Iwan parameters. To assure that the power-law behavior was accurately captured, the low-level material damping ζ_0 was subtracted from the estimated damping by visually inspecting the damping versus amplitude curve. Then, a line of the following form

$$\zeta\left(\left|Q_r\right|\right) = R\left|Q_r\right|^{\chi+1} \tag{12}$$

where R and χ are constants, was fit to the log damping versus log amplitude using least squares. Note that macro-slip was not observed in any of these simulations (and must be avoided for the modal Iwan model to retain its validity). Hence, the joint stiffness cannot be measured and so it was simply assumed to be such that the frequency of each mode shifts by 0.05 Hz in macro-slip. This and the linear natural frequency were then used to find K_{T}, and then these values were used to solve for a value of F_{S} and β such that the power law strength, R, in the Iwan model was equal to that obtained from the curve fit. In essence, the model used is equivalent to a Palmov model [14], since macro-slip is never activated. In all cases the modal Iwan model was found to fit the measured modal response very well, as illustrated for Mode 1 in Fig. 3. The modal Iwan parameters obtained for each mode are shown in Table 2.

The modal Iwan model is a SDOF model that could be integrated in response to an applied load (mapped onto the mode of interest) to compute the transient response. For example, the 100N half-sine pulse used to derive the parameters for Mode 1 was applied to its modal Iwan model and the transient response was computed using the Newmark integrator. The transient response thus computed is compared to the "measured" modal response $q_1(t)$ in Fig. 4. While the computed and "measured" responses do eventually go out of phase due to small frequency errors, the simulation captures the amplitude and frequency of the "measured" response very well over the entire range of response amplitude. Thus, we can proceed to use this modal Iwan model with confidence.

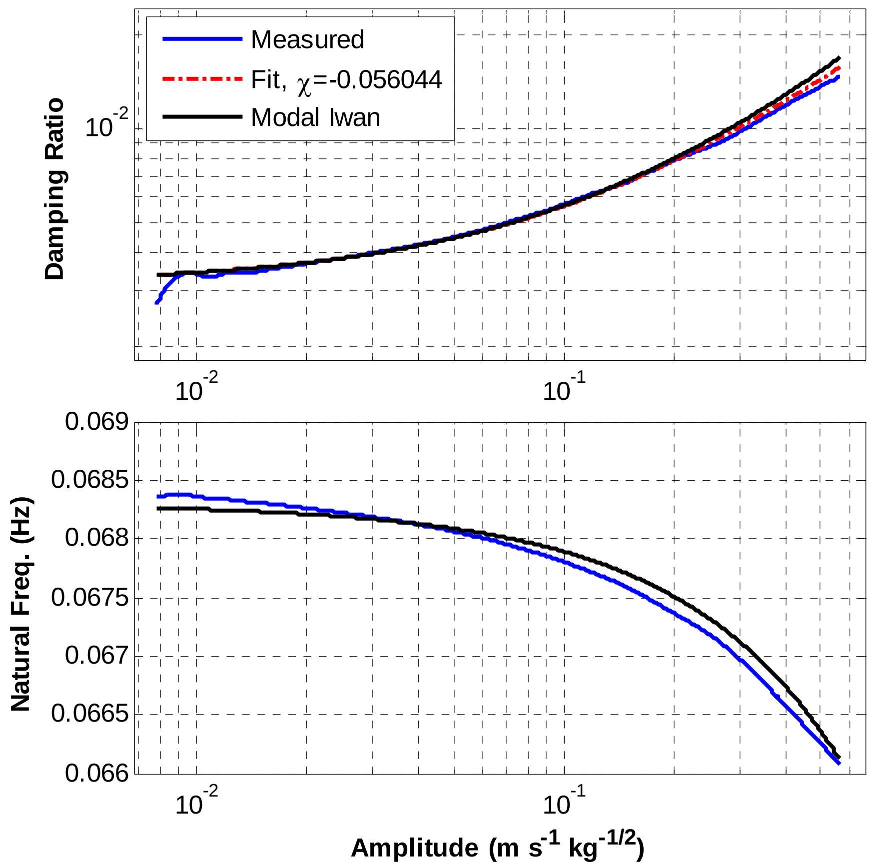

Figure 3. (**blue**) Damping ratio and natural frequency estimated using the Hilbert transform, and (**black**) those of a modal Iwan model fit to the measurements. (red dash-dot) Curve fit $\zeta\left(\left|Q_r\right|\right)=R\left|Q_r\right|^{\chi+1}$ to the damping ratio vs. amplitude, which was used to estimate the modal Iwan parameters.

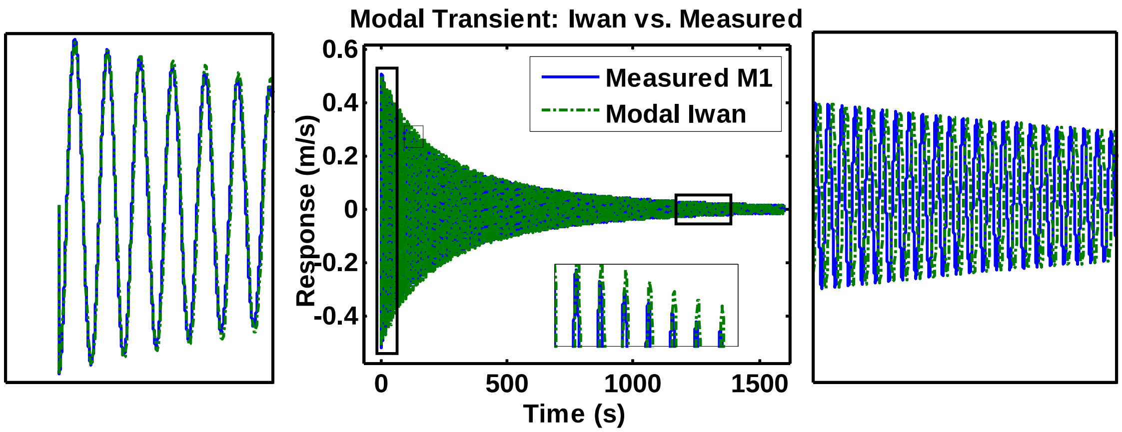

Figure 4. (**blue**) True transient response of Mode 1, $q_1(t)$, due to the half-sine impulse $\mathbf{f}(t)$. (**green dash-dot**) Estimated modal response $\hat{q}_1(t)$ computed using the modal Iwan model and the modal force $\boldsymbol{\varphi}_1^{\mathrm{T}}\mathbf{f}(t)$.

The same procedure was repeated for Modes 2 and 3 and the resulting modal Iwan parameters are shown in Table 2. For reference, the true natural frequencies and damping ratios of the linearized system are $f_{n0,true}$=[0.0686, 0.185, 0.269] Hz and $\zeta_{0,true}$=[0.00255, 0.001532, 0.001548]. The identification procedure has estimated the frequencies quite accurately, but there are errors of up to 25% in some of the damping ratios. While these errors could have been reduced by integrating longer time histories and using a finer time step to improve the accuracy of the Newmark integrator, this level of error is probably to be expected in a real experiment.

Table 2. Parameters of modal Iwan models of substructure *A*, estimated from simulated measurements. The parameters in parenthesis are not fully relevant since the modal Iwan model is only valid if the response is low enough to avoid macro-slip.

Modal Iwan Models (Substructure *A*)	(F_S)	(K_T)	χ	β	f_{n0}	ζ_0
Mode 1	**0.886**	**0.171**	**–0.023**	**0.0519**	**0.0683**	**0.0032**
Mode 2	**17.7**	**0.629**	**–0.641**	**0.132**	**0.184**	**0.00161**
Mode 3	**0.508**	**0.959**	**–0.564**	**0.000833**	**0.268**	**0.00172**

3.2 Substructuring Predictions

The substructures were assembled and the low-amplitude, linearized modal properties were calculated by solving an eigenvalue problem with the assembled mass and stiffness matrices including the linearized joint stiffnesses. The damping ratios were then calculated using the light damping approximation [11] (preserving the classical real modes) and are compared with the true values in Table 3. Because these modal properties were computed with the joints linearized, they include only the linear viscous damping that was used to represent the material damping and thus there is no effect from friction in the joints. The results show that the frequencies were accurately estimated, but the damping ratios show errors that are of a similar level as the errors in the estimates of the modal damping ratios of Substructure *A*.

Table 3. Linear natural frequencies and damping ratios predicted by substructuring.

Mode	True Freq. f_0 (Hz)	Estimated Freq. f_0 (Hz)	% Error	True ζ_0	Estimated ζ_0	% Error
1	0.04502	0.044842	**-0.40**	0.00156	0.001948	**24.92**
2	0.1287	0.12852	**-0.14**	0.001185	0.001307	**10.31**
3	0.17712	0.17629	**-0.47**	0.001402	0.001465	**4.49**
4	0.26524	0.26423	**-0.38**	0.001524	0.00171	**12.19**

The response of the assembly to a 100 N input was then computed, and the responses $x_1(t)$ and $x_5(t)$ are shown in Fig. 5. The substructuring predictions agree very well with the true transient response, both in frequency and damping. Perhaps further insight can be gained by considering the FFT of the response, projected onto each linearized mode of the assembly, as shown in Fig. 6. This shows that the substructuring predictions contain the correct frequency content for each mode, including small distortions which cause the modal responses to show slight coupling. (The modal responses shown were estimated my multiplying the responses with the inverse of the linear, low-amplitude mode shape matrix.)

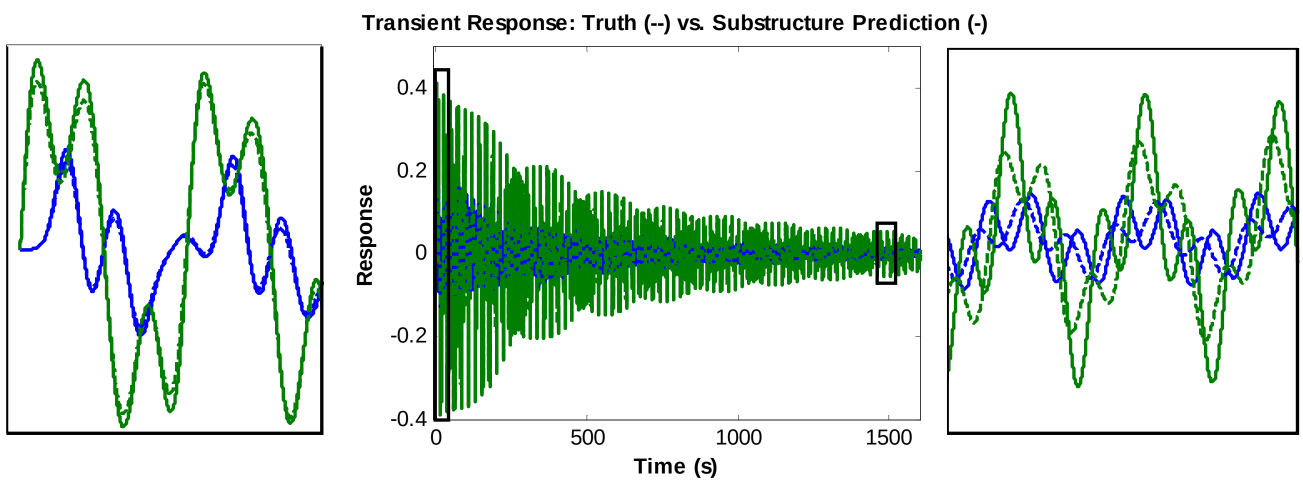

Figure 5. Transient response of the 4DOF assembly to a 100 N impulse. (**solid lines**) True response, (**dashed lines**) Substructuring prediction, using the modal Iwan model for Substructure A, (**blue**) $x_1(t)$, (**green**) $x_5(t)$. The panes on the left and right show a magnified view near the beginning and end of the response.

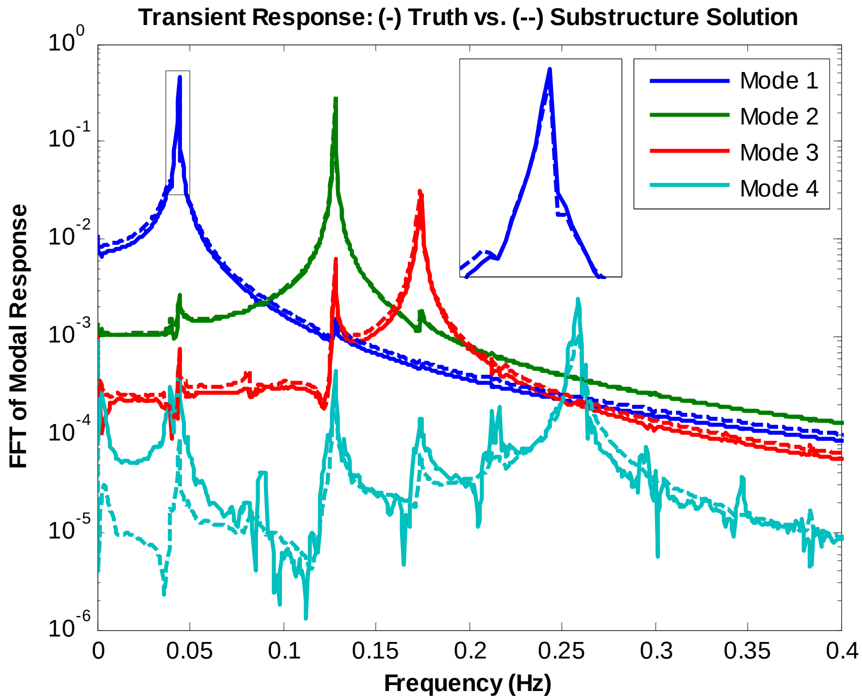

Figure 6. FFT of the transient response of the 4DOF assembly to a 100 N impulse. (**solid lines**) True response of each mode, estimated from the true response using $\mathbf{q} = \boldsymbol{\varphi}_0^{-1}\mathbf{x}$ with the linear (low amplitude) modes, (**dashed lines**) Substructuring prediction, using the modal Iwan model for Substructure A.

Most previous research, and industry practice is based on a linear approximation. Hence, it is also informative to consider whether the predictions shown above improve upon a linear approximation. An example of such a comparison is shown in Fig. 7, for an impulsive input with a 500 N amplitude. The linear approximation greatly overestimates the amplitude of the vibration, producing a response whose RMS is a factor of two larger (+99% error) than the true RMS response. Of course, the level of error incurred by using a linear model depends on the strength of the forcing. For the 100N impulsive input mentioned previously the linear model is in error by only 38%. At higher load levels the errors would be larger.

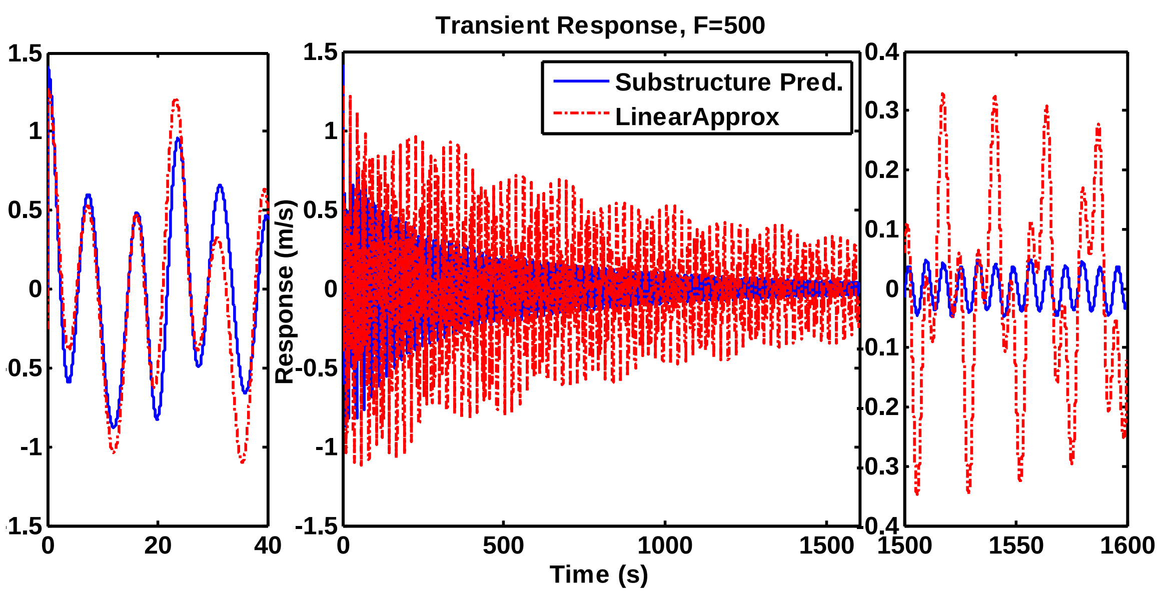

Figure 7. Transient response, $x_5(t)$, of the 4DOF assembly to a 500 N impulse. (**solid blue**) Nonlinear substructuring prediction, (**dashed red**) Response predicted by linear substructuring.

4. Conclusions

This work has proposed to model a nonlinear substructure with strong damping nonlinearities (and weak stiffness nonlinearity) due to friction at bolted interfaces using a modal approach. The linear modes are assumed to be preserved and to diagonalize the system, so that each mode's response depends only on its displacement, velocity, and on the slider states used to capture its nonlinearity. These nonlinear modal models can then be assembled using standard techniques and the equations of motion of the assembly can then be integrated using the Newmark algorithm or some other suitable integrator.

The methods were demonstrated by estimating a modal Iwan model for each mode of a 3DOF system from simulated transient response measurements due to an impulsive load. Then these modal Iwan models were used to create a nonlinear model for the substructure that was then assembled to a linear 2DOF system. The proposed approach was then used to integrate the assembled equations subject to various impulsive loadings, producing estimates of the response that were found to be quite accurate. The accuracy seemed to be primarily limited by the accuracy with which the modal Iwan model could be fit to the simulated measurements. Of course, if the forcing amplitude became so large that one of the joints exhibited significant macro-slip then the modal approximation breaks down and errors were observed (although, for brevity, no such cases were reported here).

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