

# Uncertainty in Experimental/Analytical Substructuring Predictions: A Review with Illustrative Examples

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## Abstract

Experimental-analytical substructuring methods have long been explored to expedite testing and analysis of built up systems in various fields. However, many of these efforts have failed because the substructuring calculations can be very sensitive to experimental uncertainty and truncation of the subcomponent models. This work presents a review of the literature regarding uncertainty in experimental/analytical substructuring, highlighting the phenomena that have been observed such as inherent ill-conditioning, cross axis sensitivity, uncertainty modeling and propagation and the fact that experimental measurements may exhibit phenomena that are not physically realizable. Relatively little is found in the literature regarding the last of these issues, so it is explored in more detail. A simple example problem in which one flexible beam is removed from a T-shaped structure is used to validate the theory that is presented and to explore some of these issues in more detail.

## 1 Introduction

There has been increasing interest recently in methods of coupling experimentally derived substructure models to analytical models. This is especially desirable when the structure of interest is assembled from subcomponents that are provided by an outside source, since one might not be able to change their design if that is the case. In other applications, a certain subcomponent may be difficult to model due to ill-characterized materials or intricate geometry. It may be easier to perform a dynamic test on these subcomponents and then couple that model to a simulation model for the rest of the structure to predict the behavior of the built-up system. This can reduce the cost of product development, circumventing the need to model certain subcomponents even though their dynamics are important to the performance of the built-up system.

Experimental-analytical substructuring methods have been of interest for several decades. However, although substructuring is commonplace in the analytical realm, for example the Craig-Bampton modal substructuring method [1], efforts to extend the approach to experimental substructures have not been widely adopted. Substructuring predictions can be highly sensitive to noise and small inconsistencies in experimental measurements, and the experimental model can only be obtained over a limited frequency range, so modal truncation can induce errors. In extreme cases, one may obtain completely erroneous substructuring predictions.

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This work presents a survey of the literature, summarizing several issues that can cause substructuring predictions to be highly sensitive to uncertainties. A few of those issues are explored in detail. Also, some new methods for understanding and assessing non-physical results in substructure uncoupling are presented, especially the problem of negative mass. These are important because finite element packages cannot usually to import a substructure with nonphysical parameters, and they can undermine one's physical insight. Furthermore, there is some evidence that the modes that arise due to nonphysical parameters such as negative mass can be extremely sensitive to uncertainty and may be major contributors to errors in other modes. The concepts are illustrated on a simple problem involving the removal of one beam from a T-shaped structure.

The rest of this paper is organized as follows. The following section is devoted to the literature review on substructuring uncertainty. Section 3 explores the issue of negative mass in substructure uncoupling, and in Section 3.2 those ideas are applied to a few substructures to further explore the issue. Section 4 presents the conclusions.

## **2 Literature Review: Uncertainty in Experimental/Analytical Substructuring**

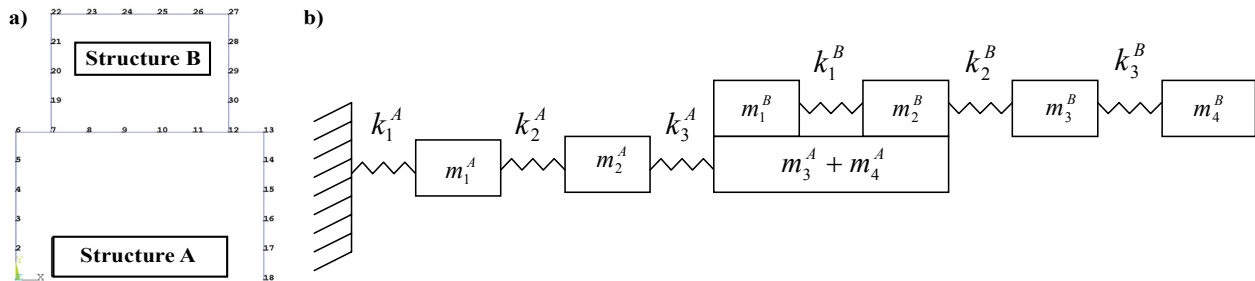
Two general classes of methods exist for predicting the response of a built up structure from subcomponent measurements, frequency response based substructuring (FBS) and modal substructuring (MS). Substructuring methods are also very closely related to structural modification (SM) methods; several methods have been presented in the literature under this name that are equivalent to the FBS and MS approaches [2-6]. Substructuring techniques were recently reviewed in detail by De Klerk et al. [7], whose nomenclature we shall adopt here. Various other names are used in the literature, especially for FBS, which is also called Admittance modeling [8, 9], Impedance coupling [3, 10, 11], etc... The MS approach predicts the modes of the built up system from the modes of the subcomponents, so it requires that one extract the modes of each subcomponent from experimental measurements. By reducing a set of measurements to the contributing modes, the quantity of data to be manipulated is greatly reduced, and one estimates the modes of the assembled system directly, so the result may be easier to interrogate. On the other hand, modal extraction can be difficult and may introduce uncertainties, so many researchers have preferred the FBS approach which uses the frequency response functions (FRFs) of the subcomponents directly to predict the FRFs of the built up system. The full FBS approach typically requires a larger set of measurements, since the FRFs for all degrees-of-freedom at each of the connection points must be captured; this may require tens of inputs and several hundred FRFs. On the other hand, one could also use FBS with reconstructed FRFs so that reciprocity can be used to construct the unmeasured FRFs. When that is done, the FBS and MS approaches should give virtually identical results, but both are still useful since they provide unique insights into the factors that can contribute to uncertainty in the substructuring predictions.

Another important distinction is between methods that couple two substructures together, and those that attempt to remove one substructure from another. The latter are termed substructure uncoupling or substructure decoupling [12]. Some of the earliest works in this area involved the removal of rigid masses from a structure [13, 14], yet recent works have made the removal of a flexible structure from another a possibility [15, 16].

### **2.1 Inherent Ill-conditioning in Substructure Uncoupling**

A few works have encountered situations in which substructure uncoupling is hypersensitive to uncertainties due to inherent ill-conditioning in the physics of the problem. For example, in his thesis, Ind reports on the problem of uncoupling one u-shaped frame from another, as shown in Figure 1a (from [17]). He shows that, for this system, one obtains completely meaningless results when attempting to remove B from the assembly (A+B) if only 1% noise is added to the FRFs. Fortunately, this type of ill conditioning has a physical basis and so it can be understood and avoided. This can be explained by the

simpler problem, shown in Figure 1b, of removing the system of masses with superscript B from those with superscript A. Clearly it would be impossible to deduce anything about  $k_1^B$  from the assembly since that spring is not at all active. This issue was called “stiff spring” ill-conditioning by Urgueira [11], and, fortunately, it can be readily avoided if one considers the issue when formulating the substructuring problem.



**Figure 1: Excerpts from Figures 4.5.1 and 5.4.6 from Ind [17] illustrating stiff-spring ill-conditioning in substructure uncoupling. In either system shown, one cannot deduce the dynamics of system B from those of the assembly because certain motions are effectively eliminated by the assembly.**

Another very different type of ill-conditioning is possible when the subcomponent to be removed is capable of resonances that do not involve any of the connection degrees of freedom [12, 16]. Traditional FBS methods require only the connection point FRFs to remove one subsystem from another, but motions that are poorly observable on that set of FRFs can still contaminate the uncoupling predictions causing spurious resonances to appear. Sjovall and Abrahamsson [16] suggested including FRFs away from the connection point in the uncoupling process in order to eliminate these internal resonances, and showed that this resolves the problem. Voormeeren & Rixen confirmed that result, presenting an analytical uncertainty analysis of that procedure, which verifies that the inclusion of internal degrees of freedom reduces the uncertainties in the predicted FRFs dramatically [18]. Working independently on a related issue, Allen, Mayes & Bergman [15] suggested using all of the measurement points on the substructure that is to be removed in order to enforce a set of modal constraints for modal substructuring; this method also avoids this type of ill-conditioning. Thanks to these works, it is now clear that one must often consider other points in addition to the connection points in order to successfully remove one substructure from another. When this issue is adequately addressed, relatively complicated subcomponents can be subtracted from a structure, as illustrated in [16] and [15].

## 2.2 Inadequate Span of the Modal Basis

Another potential source of error in substructure coupling and uncoupling arises because the subcomponent models have limited fidelity, typically describing the structure only up to some maximum frequency. Theoretically, this is not necessarily an issue for frequency based substructuring since the tails of modes above the testable bandwidth are present in the measurements, but FBS is often employed with reconstructed FRFs which may not include the residual terms. Significantly greater effort is required to identify the residual terms, and in any event they may be very weakly represented in the measurements, so one must always be aware of this issue whether FBS or MS is used. The limitations imposed by a truncated modal basis are well known for modal substructuring and a few remedies have been proposed. For example, MacNeal [19] and Rubin [20] presented methods for including the static flexibility of out of band modes in MS predictions, and this has been implemented with good success by Martinez et al. [21, 22]. Other researchers have used raw measurements in FBS, although for a much more heavily damped system, and achieved good results [23]. It is important to note that the residual terms are constants, stiffnesses between the modal model and the points of interest; they do not capture the mass of the out of

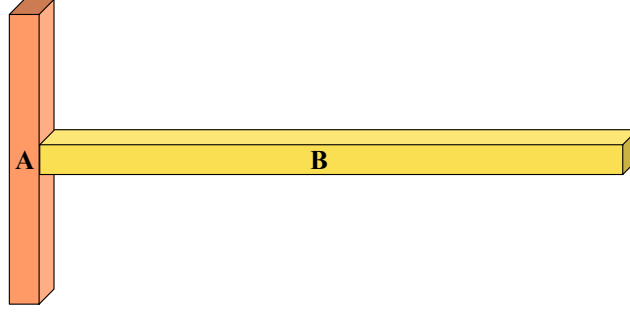
band modes. Those residual terms are typically more difficult to estimate than the modal parameters since they are constant with frequency and hence may not dominate the measurement in any frequency band as a lightly damped mode does, so is difficult to be sure that they have been captured adequately. Martinez et al studied this when joining two beams, finding that the substructuring predictions were sensitive to the rotational residual flexibilities, which were difficult to estimate; they estimated those terms by curve-fitting the displacements, reconstructing the FRFs and then using a finite difference approach to estimate the rotational FRFs from the reconstructed displacement FRFs.

One can also understand the problem of modal truncation in terms of a Ritz analysis of a structure, where a set of shape functions are presumed to form an adequate basis for the mode shapes of interest. For example, consider the problem of coupling two beams. The modes of each of the beams are easiest to identify if the beams are free, but no forces or moments act on the ends of the beams, so every vector in the basis shows zero shear and moment at the interface and hence that basis does not describe the coupled system well because shear and moments across the connection are critical. Goldenberg & Shapiro suggested that the modal basis could be improved by mass-loading the interface [24], and later studies have seen good success when using rigid masses to aid in estimating the connection point rotations [4, 13, 15]. This approach is not so straightforward to apply to three-dimensional interfaces, as illustrated by [25], but the method presented in [15] does provide some new opportunities when substructures are joined at multi-point, statically indeterminate interfaces. A somewhat similar method was also recently presented by Corus, Balmès & Nicolas [2]. They created a finite element model of the region at the interface between the substructures (or where a structural modification might be applied) and used that model to avoid measuring the rotations at the connection points and to assure that the interface was capable of the appropriate motion (e.g. motion that might not be possible otherwise due to modal truncation).

### 2.3 Analytical Studies of Uncertainty in Modal Substructuring

Because it is so preeminent in modeling, several works have sought to derive error bounds for modal substructuring predictions, especially the Craig Bampton method. Much of this work has transpired in the applied mathematics community, and so it is not well known to many structural dynamicists. One important result is the proof presented by Bourquin that shows that each mode of a one-dimensional assembly is primarily affected by modes of the subcomponents with similar natural frequencies [26]. This is somewhat intuitive for frequency based substructuring, since each mode primarily affects the FRFs near its natural frequency, but it is still an important result and merits an example.

Consider a simple substructuring problem where two beams are joined to form a T-shaped structure, as shown in Figure 2. Two-dimensional finite element models were created of each of the beams, comprised of 21 and 30 elements respectively, and the free modes of each were found, resulting in a model with 90 free modes for B and a model for A with 63 free modes. Since the modes of each substructure are known, the generalized coordinates for the system are the modal amplitudes  $\mathbf{q}_A$  and  $\mathbf{q}_B$  and the total set of degrees of freedom for the coupled system are  $\mathbf{q} = [\mathbf{q}_A^T, \mathbf{q}_B^T]^T$ . The  $m$ th natural frequency of A and the  $n$ th natural frequency of B are denoted  $\omega_m^A$  and  $\omega_n^B$ . The properties of the system used in the following were:  $\rho = 7800\text{kg/m}^3$ ,  $E = 210\text{ GPa}$ , thickness of beams = 19mm (0.75 in), width of beam = 25.4mm (1 in), length of beam A = 152.4mm (6 in), length of beam B = 304.8mm (12 in).



**Figure 2: Schematic of simple substructuring problem where beams A and B are joined forming a T-shaped structure C.**

The methods described in [27] (and similarly in [7]) were then used to enforce equal displacement and rotation at the connection point (idealized as a single point), as was done in [15]. That process involves eliminating some of the generalized coordinates to form an unconstrained set,  $\mathbf{q} = \mathbf{B}\mathbf{q}_u$ . The equations of motion for the coupled system can then be written as

$$\hat{\mathbf{M}}\ddot{\mathbf{q}}_u + \hat{\mathbf{K}}\mathbf{q}_u = \mathbf{Q} \quad (1)$$

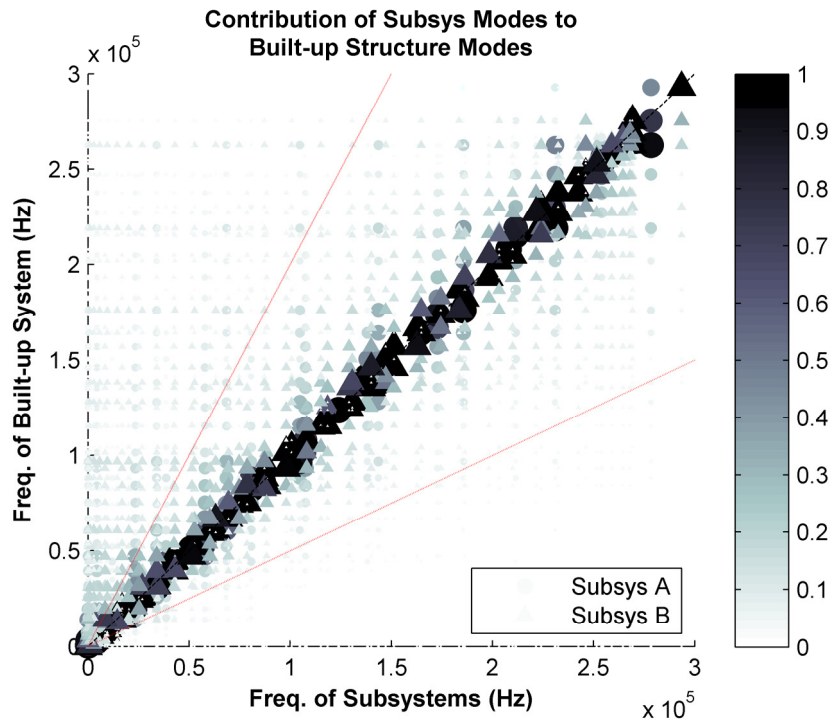
and the natural frequencies and mode shapes can be found by solving an eigenvalue problem to find the natural frequencies of the built-up system, which are denoted  $\omega_r^C$ , as well as a mode matrix,  $\Phi^C$ . That mode matrix relates the unconstrained generalized coordinates,  $\mathbf{q}_u$ , to the amplitudes of the modes of C, which we shall denote  $\mathbf{q}_C$ .

$$\mathbf{q}_u = \Phi^C \mathbf{q}_C \quad (2)$$

In each mode of the coupled system, the amplitude of each of the original degrees of freedom,  $\mathbf{q} = [\mathbf{q}_A^T, \mathbf{q}_B^T]^T$ , can be determined from  $\mathbf{q}_u$ , and a certain element of  $\Phi^C$ . Hence, one can compute the amplitudes of each of the modes in A and B as follows.

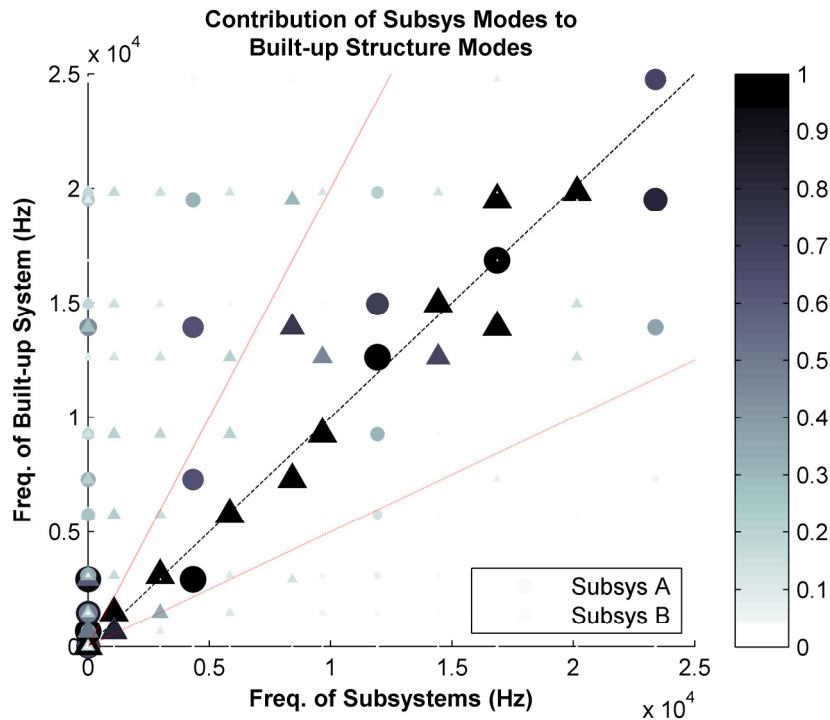
$$\begin{bmatrix} \mathbf{q}_A \\ \mathbf{q}_B \end{bmatrix} = \mathbf{B}\Phi^C \mathbf{q}_C \quad (3)$$

Denoting the  $r$ th column of  $\Phi^C$  as  $\Phi_r^C$ , the upper partition of  $\mathbf{B}\Phi_r^C$  gives the amplitudes of the modes of A in the  $r$ th mode of C and the lower partition gives the contribution of the modes of B. This is displayed graphically in Figure 3. Each column of  $\mathbf{B}\Phi_r^C$  is normalized to a maximum value of one and then each element is plotted with the vertical scale giving the natural frequency of the coupled system mode,  $\omega_r^C$ , and at a horizontal position given by the corresponding natural frequency of the subcomponentes,  $\omega_m^A$  or  $\omega_n^B$ . The size and color of each marker is proportional to the value of that element of  $\mathbf{B}\Phi_r^C$ . Circles are used for the natural frequencies of A and triangles for those of system B.



**Figure 3: Bubble plot showing the effect of each of the subsystem natural frequencies (horizontal axis) on the natural frequencies of the built-up system, C. The natural frequencies of subsystem A are shown with circles and those of B with triangles. The size and color of each marker are related to the size of the corresponding element in  $B\Phi_r$ .**

The plot shows a strong clustering along the diagonal, indicating that the modes of A and B that are nearest in frequency to the built-up system mode are the strongest contributors. The lower frequency modes of both A and B are sometimes seen to contribute somewhat, and considering the large number that contribute, their net effect is probably quite significant, while it is rare that a mode above  $2*\omega_r^C$  has a significant contribution.



**Figure 4: Zoom view on the low frequency region in Figure 3.**

It is also informative to view the same results over a much smaller frequency range. Figure 4 shows the same result but focusing on frequencies from 0 to 25 kHz. In contrast with the higher frequency region, low frequency modes seem to frequently include large contributions from the rigid body modes of the substructures. Also, one can see that when only a small number of modes are considered, there seems to be a larger spread in the frequencies of the modes that have a significant effect on each mode of C.

Many other researchers have studied substructuring seeking, for example, to derive a priori estimates of the accuracy of analytical substructuring predictions. For example, Elssel & Voss used the Raleigh quotient and a rational approximation for the eigenvalue problem to derive an error bound for MS, but they noted that “As it is often the case for a priori bounds in general problems, the error bound overestimates the true relative error by one or two orders of magnitude, ...[but] the bound cannot be improved without further assumptions.” Perhaps further work in this area will produce additional useful results.

## 2.4 Measurement Inaccuracies

Errors in FRF measurements and in the modal parameters extracted from them can arise from a number of sources, and this can have a dramatic effect on dynamic substructuring predictions. Several of these issues have been studied in the context of dynamic substructuring, as discussed in the following subsections.

### 2.4.1 Measurement Inconsistencies: To Curve-Fit or Not?

When modal tests are performed, small errors are inevitable in the measurements. For example, noise in the FRFs can make the peaks appear to shift from the true natural frequencies. When the input or responses are repositioned, as in a multi-input-single-output roving hammer test, or a multi-input-multi-output shaker test, the natural frequencies of the structure may shift slightly due to small changes in the sensor mass loading. Likewise, if the structure is even slightly nonlinear, for example due to bolted joints that might slip microscopically when the structure is loaded [28, 29], then the structure’s natural

frequencies might appear to shift from one measurement to the next. Errors in the measurements may be even more severe away from resonances, where the response may be several orders of magnitude below the peak response. At such low amplitudes, windowing errors, non-idealities in sensors and noise may have a very important effect. Considering the frequency based substructuring approach, at each frequency the FRFs of the coupled system are computed from the FRFs of the subcomponents at the same frequency, so those regions of the frequency band may become resonances of the built-up structure, greatly amplifying measurement errors.

Imregun, Robb and Ewins studied a related issue in a 1987 paper [3] for the FBS method, showing that very large errors might exist in coupled system predictions due to relatively small (~5%) errors in the subcomponent FRFs. They showed that curve fitting each individual FRF did not mitigate these errors; one had to perform a global curve fit thereby forcing a physically consistent model upon all of the measurements in order to obtain reasonable predictions. If this was not done, peaks would appear in the predicted built-up structure FRFs at many of the natural frequencies of the subcomponents. Physically, one might interpret that as meaning that some part of those subcomponent modes remained at their original natural frequencies rather than shifting to the proper values when the subcomponents were assembled. This type of argument has led many subsequent researchers to extract the modes from the subcomponent models in order to assure that the models are physically meaningful (for example, [15, 30, 31]). However, for many problems of interest the modal density is high and modal parameter extraction is not possible, so there is continued interest on FBS coupling with measured FRFs [8, 23].

One recent work explored this type of sensitivity, comparing the perspective provided by modal substructuring with that of FBS. Allen & Miller simulated mass modifications to a very simple structure in which two modes had been mistakenly identified where only one actually existed, so the mode was artificially split into two [32]. Surprisingly, they achieved good predictions of the modified structure's FRFs so long as the identified modal model accurately fit the FRFs of the base structure at the point(s) where the mass was attached.

#### **2.4.2 Cross-Axis Sensitivity**

The piezoelectric accelerometers that are most frequently used for modal tests are known to erroneously register vibration from directions other than the intended one, a phenomenon called cross-axis sensitivity. Likewise, there is always some uncertainty regarding the directions of the inputs applied to the structure, which causes the measurements to greatly overestimate the degree to which the structure responds to forces in the perpendicular direction. Even non-contact transducers such as laser vibrometers are susceptible to contamination due to uncertainty in the orientation of the laser and due to laser speckle noise [33-35]. Cross-axis errors can have an important effect on substructuring predictions. For example, they are thought to be responsible for artificially high levels of axial motion in the bending response of a beam structure that was studied in [15, 30]. In other cases the effect might be more severe. Nicgorski & Avitabile highlighted this issue in a recent work and suggested that, rather than performing substructuring with the contaminated measurements directly, one could instead use the measurements to tune an approximate FEA model for the substructure in order to eliminate these errors, a technique that they call VIKING (Variability Improvement of Key Inaccurate Node Groups) [36].

#### **2.4.3 Nonphysical Measurements**

Errors in FRF measurements can sometimes be detected when they cause the FRFs to exhibit phenomena that are not possible for a physical system. For example, one can show that the imaginary part of an accelerance frequency response function must always be positive at the drive point since the mode shapes appear as squared quantities at the drive point. Carne & Dohrman found that spurious peaks can appear in substructuring predictions if this feature is violated and presented a technique called DeComposition Data (DCD) filtering to address it [8]. Their method was found to successfully remove several spurious peaks from FBS predictions for a quite complicated system.



A somewhat similar problem can arise in the modal domain when one obtains a system model whose mass matrix is not positive definite. Kanda, *et al.* observed this in an early work when removing rigid masses from a subcomponent model [4]. They found that the mass matrix for the subcomponent's modal model was sometimes not positive definite after removing the masses. They attributed this non-physical result to errors in the modal scale factors that were determined experimentally and adjusted them until the mass matrix became positive definite as expected. The authors have observed this phenomenon as well when subtracting one flexible structure from another [15, 30]. This issue merits further study and will be explored in detail in Section 3.

#### 2.4.4 Other Uncertainties and Uncertainty Propagation

Several other types of uncertainties have also been studied that do not fit into the categories listed above. De Klerk recently studied the effect of small errors in the orientation and location of hammer inputs in a roving hammer test, showing that very small errors can sometimes have an important effect on FBS predictions [37]. Zhang *et al.* [14] noted that the scale of the modes derived from test may be significantly in error and suggested that one could reduce the error could by applying mass-modifications, measuring the shift in each natural frequency and adjusting the mode scaling until the model correctly predicts the observed frequency shift. Their approach was found to work well, although complications arise when the structure has modes with close natural frequencies.

Recently, a few different studies have sought to model the uncertainty in a substructure model and to use that to predict the expected uncertainty in the output. To succeed, one must first estimate the uncertainties in the substructure model. The random errors in FRF measurements can be readily quantified using the coherence function, and Verboven *et al.* have devised a method for estimating the uncertainties in a structure's natural frequencies, damping ratios and mode shapes from those [38, 39]. Unfortunately, that class of random error may represent a relatively small fraction of the overall uncertainty, which arises from each of the effects just mentioned as well as from random noise. Cafeo *et al.* [40] studied this in detail by replicating modal tests on a series of automobiles. They found that a Gaussian distribution with outliers described the measured uncertainties well. Other works are needed to expand upon this. In the meantime, uncertainty propagation can still be very helpful even if the substructure uncertainties must be assumed or worst case bounds used.

The most straightforward way to study the effect of uncertainty is through Monte Carlo Simulation [41], where the substructuring predictions are simply repeated using different values for the uncertain system parameters. One can then compute the statistics of the response and robustly quantify the effect of uncertainty on the built up system's response. For example, Bergman *et al.* used this method to compare two modal substructuring techniques in [42]. The substructure models employed in experimental analytical substructuring are usually quite small, so it is typically relatively inexpensive to perform hundreds of Monte Carlo trials in order to assess uncertainty. When the computational burden becomes too great, other approximate approaches from the uncertainty quantification literature can be adopted, many of which are described and discussed in [43, 44]. However each method requires different assumptions regarding the input-output relationship, and one generally does not know a priori whether those assumptions are warranted.

Approximate methods also exist that circumvent the computational burden required by MCS and provide considerable insight. One of the simplest approaches is based on a linear approximation for the FRF sensitivity, dubbed the "Embedded Sensitivity" approach by Yang, Adams *et al.* [6, 45, 46]. Under mild assumptions, one can compute the sensitivity of the FRFs to small mass or stiffness modifications using the measured FRFs alone. For example, they give the following formula for the change in the frequency response function for output  $j$  and input  $k$  due to the addition of a small mass,  $\Delta m_n$ , at node  $n$ .

$$\Delta H_{jk}(\omega) = \omega^2 H_{jn} H_{kn} \Delta m_n \quad (4)$$

This reveals that the change in the FRF depends on the product of two other FRFs, which can easily be measured and used to compute the sensitivity. This approach is convenient and provides considerable

insight, but one should bear in mind that the FRF is a nonlinear function of the modal parameters, so the linear sensitivity approximation may only be valid for a very small modification.

The recent work by Voormeeren, de Klerk & Rixen [47, 48] expands on this concept considerably. They use the linear terms in a Taylor series expansion of the substructuring equations to compute the uncertainties in the FRFs of the built-up structure from those of the subcomponent FRFs. This reveals how the uncertainties in each FRF affect the assembly. For example, they found that uncertainties in the connection point FRFs cause corresponding uncertainty in all of the FRFs of the assembled system, while, uncertainties away from the connection point do not affect the other FRFs [47]. Also, as was seen in Eq. (4), the FRFs of the built-up system were often most sensitive to uncertainty near the natural frequencies of the individual subcomponents. That work was recently extended to consider substructure uncoupling as well, as mentioned previously [18].

### 3 Negative Mass after Substructure Uncoupling

As discussed in Section 2.4.3, substructure uncoupling can produce system models that have non-physical parameters such as a mass matrix with non-positive definite mass. If the mass matrix for a structure has negative eigenvalues, this would imply that a certain velocity of the structure could produce negative kinetic energy. The authors have found that this situation can also cause the substructure to have spurious natural frequencies that may contaminate substructuring predictions. For these reasons, some findings in this area are described below. The modal substructuring approach, presented in [15], is adopted in all of the following.

#### 3.1 Theoretical development

Consider the case where a fixture, system A, is to be removed from system C, which is comprised of substructure B and fixture A. The assembly, system C, is tested and its modal parameters identified, but those of the individual subcomponents are unknown. Using the approach in [15, 30, 31], each substructure is represented by its free modes as follows

$$\begin{aligned} \mathbf{I}\ddot{\mathbf{q}}_A + \left[ \begin{array}{c} 2\zeta_r \omega_r \\ \omega_r^2 \end{array} \right]_A \dot{\mathbf{q}}_A + \left[ \begin{array}{c} \omega_r^2 \\ \omega_r^2 \end{array} \right]_A \mathbf{q}_A = \mathbf{\Phi}_A^T \mathbf{F}_A \\ \mathbf{y}_A = \mathbf{\Phi}_A \mathbf{q}_A \end{aligned} \quad (5)$$

where  $\mathbf{I}$  denotes an  $N_A \times N_A$  identity matrix,  $\left[ \begin{array}{c} 2\zeta_r \omega_r \\ \omega_r^2 \end{array} \right]_A$  an  $N_A \times N_A$  diagonal matrix of modal damping constants, and  $\left[ \begin{array}{c} \omega_r^2 \\ \omega_r^2 \end{array} \right]_A$  an  $N_A \times N_A$  diagonal matrix of modal natural frequencies squared. A similar description can be written for C. Using the method in [15], one can remove A from C by making the identity (mass), damping and stiffness matrices negative and then coupling A to C using modal constraints. Specifically, suppose that the response has been measured on the fixture at a set of points denoted  $m$ . One could constrain A to C at all of the points  $m$  using,

$$\mathbf{y}_{Am} = \mathbf{y}_{Cm} \quad (6)$$

but that could lead to ill conditioning if there are any errors in the measurements at the experimental points  $y_{Cm}$ . Hence, Allen, Mayes & Bergman [15] suggested relaxing the constraint by premultiplying the constraint above by the pseudo inverse of the fixture mode shape matrix, which is assumed to be full column rank. The constraint can then be written as

$$\left[ \mathbf{I} \quad -\mathbf{\Phi}_{Am}^+ \mathbf{\Phi}_{Cm} \right] \begin{Bmatrix} \mathbf{q}_A \\ \mathbf{q}_C \end{Bmatrix} = \mathbf{0} \quad (7)$$

and one can proceed to define a set of unconstrained generalized coordinates  $\mathbf{q}_u$ , which are related to the full set of generalized coordinates  $\mathbf{q} = [\mathbf{q}_A^T, \mathbf{q}_C^T]^T$  as  $\mathbf{q} = \mathbf{B}\mathbf{q}_u$ . To satisfy the constraints, the matrix  $\mathbf{B}$  must

be in the null space of the matrix on the left in eq. (7). One can then use  $\mathbf{B}$  to compute the unconstrained equations of motion of the system, as described in [15] or [7, 27].

In order to address the issue of nonphysical parameters, consider selecting the modal degrees of freedom of C as the unconstrained generalized coordinates, which can be done using the following choice for  $\mathbf{B}$

$$\begin{Bmatrix} \mathbf{q}_A \\ \mathbf{q}_C \end{Bmatrix} = \mathbf{B} \mathbf{q}_C \quad (8)$$

with

$$\mathbf{B} = \begin{bmatrix} \Phi_{Am}^+ \Phi_{Cm} \\ \mathbf{I} \end{bmatrix} \quad (9)$$

One can then write the estimate for the mass matrix of the B system (recall that  $\mathbf{B} = \mathbf{C} - \mathbf{A}$ , or B is the result of coupling C to a negative model for A), as follows.

$$\begin{aligned} \mathbf{M}_B &= \mathbf{I}_C - \tau^T \mathbf{I}_A \tau \\ \tau &= \Phi_{Am}^+ \Phi_{Cm} \end{aligned} \quad (10)$$

Hence, the eigenvalues of  $\mathbf{M}_B$  are one minus each of the eigenvalues of  $\mathbf{M}_{\hat{A}} = \tau^T \tau$ . Note that

$\mathbf{P}_{Am} = \Phi_{Am} \Phi_{Am}^+$  is an orthogonal projector onto the column space of  $\Phi_{Am}$ , so  $\tau$  is related to the orthogonal projection of C's modes onto the column space of  $\Phi_{Am}$ . One can show that the matrix  $\mathbf{M}_{\hat{A}}$  is exactly the mass matrix that one would obtain by reducing the mass matrix of an FEA model for the fixture to the measurement points,  $m$ , using a SEREP reduction [10, 49]. Alternatively,  $\mathbf{M}_{\hat{A}}$  can be thought of as the mass matrix of a Ritz model for system A, when the modes of C are used as basis vectors. In any event, if any of the eigenvalues of  $\mathbf{M}_{\hat{A}}$  are greater than one, then  $\mathbf{M}_B$  will have negative eigenvalues and will not be a physically realizable mass matrix.

The concepts above can be used to compute the contribution of each of the modes of system C (or, more specifically, the mass of A that is projected onto each of the modes of C) to the eigenvalues of the mass matrix. Adapting the Effective Independence (EfI) approach [50] to this problem, one first forms  $\mathbf{Q} = \tau \tau^T$ . Then let  $\mathbf{V}$  denote the matrix of mode vectors of  $\mathbf{Q}$  and let  $\mathbf{L}$  be a diagonal matrix containing the corresponding eigenvalues. One can then form  $\gamma = (\tau \mathbf{V}) \cdot^2$ , where  $\cdot^2$  denotes a term-by-term square of  $(\cdot)$ . The result is a matrix  $\gamma$  whose  $m,n$ th element gives the contribution of the  $m$ th mode of system C to the  $n$ th eigenvalue of  $\mathbf{M}_{\hat{A}}$ . One can readily locate the modes of C that carry the excess mass, which is responsible for the negative eigenvalues in  $\mathbf{M}_B$ .

### 3.2 Application to Simple MS Uncoupling Problem

Consider again the system pictured in Figure 2, only now suppose that the assembly  $C=(A+B)$  exists as experimental hardware. A modal model of Fixture A is available, and we desire to use it to remove the effects of A from C using modal substructure uncoupling. Modal constraints [15] will be used between the analytical model of fixture A, denoted  $A_{an}$  and the experimental fixture  $A_e$ , which is part of C. Only the in-plane motion is considered.

The first fifteen modes of C are used in the uncoupling, corresponding to a modal test in which all modes out to 20kHz have been extracted. The natural frequencies of the elastic modes are shown in Table 1. The lower modes show the horizontal beam bending while the vertical one (fixture A) undergoes rigid body rotation. Some of the higher frequency modes show the horizontal beam vibrating axially as the fixture bends. The first six free-modes of A are used in the fixture model, three of which are rigid body modes, the 4th and 5th involve bending of beam A and mode 6 involves axial motion of beam A. Six modal

constraints are used to join A to C. Displacement in both the axial and bending directions at all 21 nodes of the finite element model are used in forming the modal constraints, although in an experiment one would likely not have such a detailed set of measurements. The rotations at those nodes are not used, since one cannot usually measure them in practice.

The table below also shows the natural frequencies of the B system estimated by the MS uncoupling procedure. The actual FEA natural frequencies of the B system are also shown as well as the percent difference. All of the MS predicted natural frequencies below 17kHz are very accurately predicted, having less than 3% error. However, the MS procedure returns three natural frequencies which are purely imaginary and do not correspond to any of the analytical natural frequencies. These modes involve motion primarily on subsystem A; the motion is three orders of magnitude smaller on component B in each of these modes. A few FRFs of the B system were reconstructed (not shown here) in the axial and bending directions, and they were seen to overlay the analytical FRFs out to 17kHz, confirming that the spurious modes did not have a large effect on the FRFs.

Mode	Subcomponents		System B		
	$f_{n,A}$	$f_{n,C}$	MS est. $f_{n,B}$	Actual $f_{n,B}$	% Error
4	4326.5	652.3	1083.3	1081.6	0.2%
5	11926.4	1453.6	0+i*1334.2	-	-
6	16853.0	2924.7	2996.4	2981.6	0.5%
7	-	3090.5	5903.3	5845.1	1.0%
8	-	5751.6	8421.5	8422.1	0.0%
9	-	7285.6	0+i*9157.5	-	-
10	-	9251.0	9824.0	9662.5	1.7%
11	-	12615.3	14832.6	14434.8	2.8%
12	-	13919.7	16965.1	16868.9	0.6%
13	-	14950.0	18066.0	20162.6	-10.4%
14	-	16853.0	0+i*20213	25365.3	-
15	-	19507.0	33706.2	26847.1	25.5%

**Table 1: Elastic Natural Frequencies of Subsystems C and A, and those that result from using MS uncoupling to compute B=C-A. The actual natural frequencies of the FEA model for B are also shown.**

The eigenvalues of the mass matrix of B that was estimated with the MS procedure were found and the lowest five were: -0.012, -0.00038, 3.9e-016, 0.0077, 0.074. Two of these are negative and one is zero, indicating that the model for B is not physically realizable, although one should bear in mind that the negative eigenvalues are only very slightly below zero, so this model might represent a good approximation for the dynamics of the system even though these parameters are not physically realizable. Even then, a model such as this is incompatible with certain solvers in FEA packages (which require positive definite mass), so one would like to avoid this.

In order to investigate this further, the source of these negative eigenvalues was investigated using the methods outlined above. The matrix  $\tau$  was formed and  $\mathbf{M}_{\hat{A}}$  was found to have two eigenvalues that were slightly greater than one and a third that was almost exactly equal to one. The EFI procedure was used to compute the contribution of each of the modes of C to the eigenvalues of the mass matrix. The dominant contributors to those eigenvalues are shown in the table below. Each column heading gives the eigenvalue of  $\mathbf{M}_{\hat{A}}$  (recall that the eigenvalues of  $\mathbf{M}_B$  are 1- the eigenvalues of  $\mathbf{M}_{\hat{A}}$ ), and the values shown are the contribution of each mode to that eigenvalue. Contributions below 0.01 have been shown with zeros to improve readability.

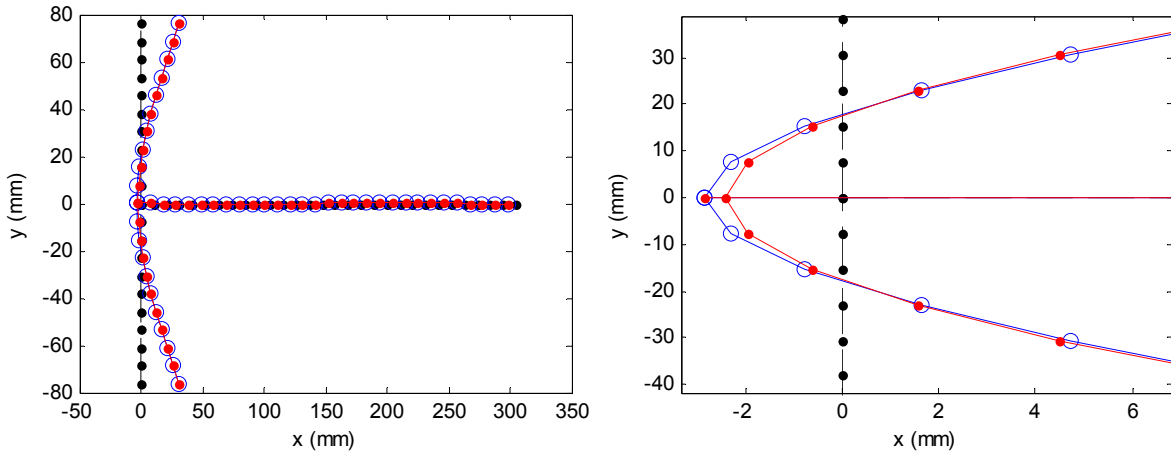
Mode of C	Contribution to Eigenvalues of $\mathbf{M}_{\hat{A}}$		
	$\lambda_A = 1.012$	$\lambda_A = 1.00038$	$\lambda_A = 1$
14	0	0	1
6	0.77	0	0
5	0	0.38	0

4	0	0.37	0
3	0.08	0.01	0
7	0	0.08	0
9	0.07	0	0
11	0	0.06	0
2	0.02	0.03	0
12	0.04	0	0

**Table 2: Contribution of each mode of C to the eigenvalues of  $\mathbf{M}_{\hat{A}}$ .**

A few interesting observations can be made. First, mode 14 is entirely responsible for the zero eigenvalue in  $\mathbf{M}_B$ . Mode 14 was found to involve purely axial motion of beam A. A corresponding mode exists in subsystem A, with the exact same natural frequency. That mode was completely unaltered when A was joined to C since it has a node at the connection point, so essentially the same mode exists in both A and C. The zero eigenvalue in  $\mathbf{M}_B$  apparently comes about because this mode's mass is entirely removed from C by the substructure uncoupling process.

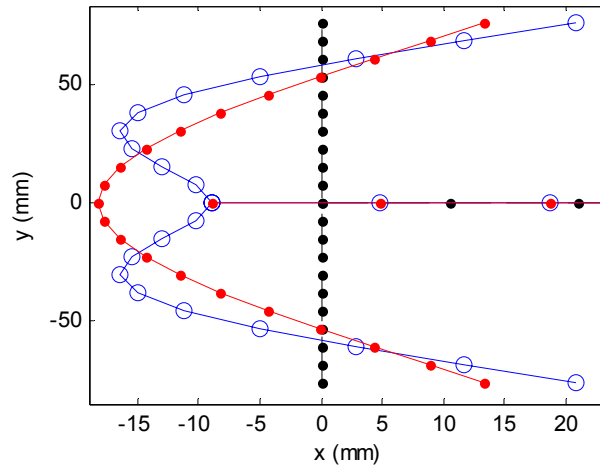
The other two eigenvalues are more difficult to interpret. The table shows that the sixth mode of C is the dominant contributor to the first negative eigenvalue, contributing 0.77 of the total value of 1.012. Mode 6, shown with a blue line and open circles in Figure 5, involves axial motion of beam B and bending motion of beam A. The lower modes of C all exhibit bending motion of B with beam A undergoing approximately rigid body rotation, so it seems that this mode carries a significant proportion of the mass associated with bending motion of the fixture, and this mass must be removed to accurately predict the natural frequencies of B. To diagnose the situation further, the orthogonal projection of C's motion onto the space of A's modes was computed as  $\mathbf{y}_{Cm,p} = \mathbf{P}_{Am} \Phi_{Cm}$ , and that motion is also shown in Figure 5 with a red line and with dots at each of the node points. The zoom view shows that the reconstructed motion matches the true motion very well; the maximum difference between the two is 1.4%. Hence, it seems that this mode's contribution to  $\mathbf{M}_{\hat{A}}$  is physical and represents mass that should be removed.



**Figure 5: Shape of Mode 6 of system C: (black/dots) undeformed structure, (blue/circles) mode of C, (red/dots) projection of C onto the free modes of A.**

The other mode shapes were also viewed, revealing that modes 9 and 12 also involve axial motion of beam B and bending of beam A. The maximum errors in projecting those modes onto the fixture's motion were 9.4% and 44.6% respectively. Mode 12's shape is shown in Figure 6 with a blue line. The plot reveals that beam A undergoes significant bending in this mode, bending which appears to be a combination of the 1<sup>st</sup> and 3<sup>rd</sup> bending modes of beam A. However, the model for beam A only included the first two bending modes of A (and one axial mode). When the actual motion (blue line) is projected onto the space spanned by the modes of A, shown with a red line, a significant discrepancy appears. This confirms that the modal basis of the fixture is inadequate to describe mode 12's motion. Because the

fixture model does not contain the third bending mode, it must attribute the third-mode motion to other modes, and so one is likely to overestimate the contribution of the first bending mode of the fixture to in Mode 12's motion. This might explain why too much mass is removed from the system when the fixture model is subtracted, resulting in negative eigenvalues in the estimated mass matrix. The first eigenvalue of  $\mathbf{M}_{\hat{A}}$  is 1.012, and the table shows that mode 12 contributes 0.04 to it, so if mode 12 were not present then this eigenvalue would reduce to 0.972 and the corresponding eigenvalue of  $\mathbf{M}_B$  would no longer be negative. It is also interesting to note that a discontinuity appears in the projection of the 12<sup>th</sup> mode shape onto the space of the free modes of A; the projection (red dots in Figure 6) shows a large gap between the center of beam A and the end of beam B.



**Figure 6: Shape of Mode 12 of system C: (black/dots) undeformed structure, (blue/circles) mode of C, (red/dots) projection of C onto the free modes of A.**

Further exploration revealed that discrepancy in Figure 6 could be reduced greatly by increasing the number of modes in system A to seven. When that was done, the maximum discrepancy between the actual mode 12 and its projection onto A's basis,  $\mathbf{y}_{Cm,P} = \mathbf{P}_{Am} \Phi_{Cm}$ , was found to reduce from 44.6% to 6.2%. The substructuring calculations were repeated and the corresponding negative eigenvalue of  $\mathbf{M}_B$  had disappeared (although the other remained).

Another alternative would be to reduce the number of modes used in C so that the six-mode model for A would adequately span the observed motion of C. This was accomplished by using six modes for system A and eleven modes for system C. The mass matrix that this model predicted was positive definite with the smallest eigenvalue being 0.00034. However, since fewer modes were used for C, the model obtained for B was only accurate up to 14kHz, whereas the FRFs were accurately reconstructed out to 17kHz when 15 modes were used for C.

## 4 Conclusions

This work presented a summary of the literature relating to uncertainty in experimental-analytical substructuring predictions. Analytical studies of substructuring were cited, and an example illustrated that the modes of an assembled system are most strongly affected by subcomponent modes that have similar natural frequencies. Some of the works cited show that significant errors can afflict substructuring predictions, but fortunately, many of the most severe errors have a physical basis, such as the stiff-spring ill conditioning mentioned by Dr. Ind, so they can be avoided through careful design of the fixturing that is to be removed. Numerous other sources of error were cited that are more difficult to quantify, so further work is warranted in those areas.

A few papers were cited which found that non-physical results can be obtained after substructure uncoupling, particularly when one substructure is subtracted from another. This phenomenon was studied analytically and a method was presented based on Effective Independence that allows one to determine which modes are contributing to each negative eigenvalue in the uncoupled system's mass matrix. An example was presented that showed that sometimes zero or slightly negative mass can be physically meaningful (i.e. in the case of the 14<sup>th</sup> mode of C, which was completely removed), while in other cases negative mass seems to arise when system A's modal basis is not adequate to span the motion observed on C.

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