Using NNMs to Evaluate Reduced Order Models of Curved Beam

Christopher I. VanDamme[1] and Matthew S. Allen[2]

[1]Graduate Student; e-mail: cvandamme@wisc.edu

[2]Associate Professor; e-mail: matt.allen@wisc.edu

University of Wisconsin-Madison: Department of Engineering Physics

1500 Engineering Drive

Madison, Wi 53706

Abstract

Reduced order modeling of geometrically nonlinear structures has gained significant attention over the recent years due to its ability to reduce detailed finite element models into computationally efficient yet accurate models. The reduced order models (ROMs) generated can be used to compute the nonlinear normal modes (NNMs) of a structure which serve as a metric to determine the accuracy of the reduced order models; if the NNMs are reproduced accurately then the response to various inputs can also be shown to be accurate. A majority of work has been focused on flat structures while relatively few works have focused on curved structures, those that have, suggested that these structures may introduce complexities to the system. This paper will provide a criteria for selecting basis vectors and load amplitudes for generating accurate ROMs for a curved beam, where softening nonlinearity and snap through are significant. Two methods will be considered: implicit condensation and expansion (ICE) and enforced displacements (ED). A framework for mode selection, load scaling factors and model sensitivity testing will be presented in order to accurately compute the NNMs.

Keywords: Reduced Order Modeling, Geometric Nonlinearities, Nonlinear Normal Modes, Finite Element Analysis, Structural Dynamics

1 Introduction

Future hypersonic vehicles may experience high thermal and acoustic loading due to extreme flow fields and engine noise [1]. These extreme loading conditions cause thin walled skin panels to vibrate nonlinearly, sometimes even about a buckled equilibrium point [2]. Linear approximations are insufficiently accurate in being able to predict the geometrically nonlinear response of the panels which leads to an over conservative design of the system [2]. Modeling is performed using the finite element method because the discretization capabilities allows for complicated geometries to be accurately modeled. The finite element models of these structures, in order to account for the complex geometries, may contain millions of degrees-of-freedom (DOF) resulting in millions of equations of motion of the system that need to be solved to find the dynamic response of the system. Computational methods are available to calculate the nonlinear response of the system to such loads using either implicit or explicit time integration of the equations of motion. The problem with explicit time integration schemes is that in order to achieve a geometrically accurate model some elements of the system must be significantly small resulting in small time step requirements. To accurately predict the response of the structure, the equations must be integrated over many cycles of the lowest frequency mode for a large set of loading conditions to account for the random loading environment the vehicles experience. This, coupled with the small time steps, cause the analysis to be prohibitively expensive. For example, a 96,000 DOF finite element model of a multi-bay aircraft panel required 6 days of computation on a quad-core computer for 100,000 time steps [3]. Analysis cycles of this length are prohibitive to the design of structures experiencing geometric nonlinearities.

Nonlinear Reduced Order Models (NLROMs) provide a method to decrease the number of DOF by approximating the full finite element model resulting in fewer equations of motion to solve. The ability to create accurate NLROMs of the discretized finite element model decreases the computational cost of the analysis resulting in decreased design cycle time. However, it is not necessarily straightforward to create an accurate ROM. An analyst must determine how many modes to include, and since each mode comes with substantial cost, the analyst should include as few as possible. The analyst must also determine how large of loads to apply to estimate the ROM. Furthermore, once the ROM is created, it comprises thousands of coefficients so it is difficult to tell what has been captured in the ROM. In the past, the static or time response of the ROMs have typically been compared with simulations from the full order model which are very expensive to obtain. Kuether & Allen recently proposed an alternative, in which the NNMs of the ROMs are compared to understand what dynamics the ROM has captured and to assess its quality. This paper tests the methodology developed by Kuether and Allen [4]. The validation of the ROMs are measured by evaluating the convergence of computed NNMs, which provide great insight into the free, forced and random response of structures [5].

In the linear domain a ROM must account for the range of loading frequencies the structure will encounter. In essence, the ROM only needs to be validated in terms of frequency; if the frequency range of the ROM encompasses the loading frequencies, then one can expect accurate predictions of the response. When geometric nonlinearities are encountered the response becomes amplitude dependent, adding a second dimension that must be considered. The ROM must now account for both the frequency and amplitude range of the loadings the structure may encounter. NNMs account for both frequency and loading amplitude, represented as energy of the system, typically plotted as a Frequency Energy Plot (FEP). A FEP represents the NNM with the structure's computed periods on the vertical scale and energy (potential + kinetic) on a logarithmic horizontal scale.

In previous works, Kuether & Allen compared the NNMs of the ROMs with NNMs computed directly from the full FEM. The latter are quite expensive to compute, so this work will instead follow the suggestion in [4]. This work will not compute the full finite element model NNM because in order for time efficient design and analysis cycles, the full model NNM would not be computed. The analyst will need to validate the accuracy of the ROM without a truth solution. This work will apply these tools to a structure with curvature, whose nonlinear dynamics are far more complicated.

Two methods of creating ROMs, Enforced Displacements (ED) and Implicit Condensation and Expansion (ICE), will be evaluated with NNMs. The two methods presented have been given large attention over the past few years for flat structures, this paper will study the two methods of model reduction on a curved structure, which is more prevalent in real designs. The curvature of the structure provides increased complexity of the problem due to strong modal coupling, presence of snap through and less deterministic load case factors.

2 Theory

The linear elastic finite element model equation of motion for N degree-of-freedom (DOF) system with no damping can be written as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t) \tag{1}$$

where $\mathbf{M}$ is the $N \times N$ mass matrix, $\mathbf{K}$ is the $N \times N$ linear stiffness matrix and $\mathbf{f}(t)$ is the $N \times 1$ external force vector. The displacement and acceleration vectors $\mathbf{x}$ and $\ddot{\mathbf{x}}$ are both $N \times 1$ vectors. This equation is valid for small displacement approximations and assumes no nonlinearities within the model. In order to account for geometric nonlinearities within the finite element model Eq. (1) must be altered to include the nonlinear restoring force vector, $\mathbf{f}_{NL}$, to form the nonlinear equation of motion of the form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{f}_{NL}(\mathbf{x}) = \mathbf{f}(t) \tag{2}$$

where $\mathbf{f}_{NL}$ is a $N \times 1$ vector. The creation of the ROM follows similarly from that of the linear modal ROM where the reduction basis is a set of linear modes of the system The linear modes of the system are found by solving the eigenvalue problem of the linear system shown in Eq. (1)

$$\left(\mathbf{K} - \omega^2\mathbf{M}\right)\phi_\mathbf{r} = \mathbf{0} \tag{3}$$

which can be solved to find the coordinate reduction equation represented as

$$\mathbf{x}(t) = \Phi_m \mathbf{q}(t) \tag{4}$$

where Φ_m is the $N \times m$ the mass normalized mode matrix comprised of the mode vectors, ϕ_r, and $\mathbf{q}$ is the $m \times 1$ vector of time-dependent modal displacements. The motivation for representing the linear equation of motion in this fashion is that the modal coordinates and mode shape matrix are able to accurately predict the response of the system while reducing the number of degrees of the freedom since $m \ll N$.

When the linear equation of motion in Eq. (1) is premultiplied by Φ^T, where ^{T} is the transpose operator, the r^{th} modal equation becomes uncoupled and is represented as

$$\ddot{q}_r + \omega_r^2 q = \phi_r^\mathbf{T}\mathbf{f}(t) \tag{5}$$

When the same coordinate transformation is applied to the nonlinear equations of motion, Eq. (2), for the r^{th} modal equation becomes

$$\ddot{q}_r + \omega_r^2 q + \theta_r(q_1, q_2, ..., q_m) = \phi_r^{\mathbf{T}} \mathbf{f(t)} \tag{6}$$

where the nonlinear restoring force, θ_r , is represented as a function of modal displacements by

$$\theta_r(\mathbf{q}) = \phi_r^{\mathbf{T}} \mathbf{f}_{NL}(\Phi_m \mathbf{q}) \tag{7}$$

In [6,7] it was shown that the restoring forces for a linear elastic system with geometric nonlinearities can be well approximated as

$$\theta_r(\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_m) = \sum_{i=1}^{m}\sum_{j=1}^{m} \mathbf{B}_r(i,j)\mathbf{q}_i\mathbf{q}_j + \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} \mathbf{A}_r(i,j,k)\mathbf{q}_i\mathbf{q}_j\mathbf{q}_k \tag{8}$$

where the nonlinear restoring force is a function of quadratic and cubic polynomials with coefficients B_r and A_r respectively for the r^{th} nonlinear modal equation. These coefficients are computed differently depending on whether using Enforced Displacements or Implicit Condensation and Expansion as described below.

2.1 Enforced Displacement (ED)

The enforced displacements procedure applies a set of known displacements in the shape(s) of the linear modes. The general form of the applied displacements for a basis set with multiple modes is described as

$$\mathbf{X}_i = \phi_1\hat{\mathbf{q}}_1 + \phi_2\hat{\mathbf{q}}_2 + \ldots + \phi_m\hat{\mathbf{q}}_m \tag{9}$$

where $\mathbf{q}_r$ is the scaling factor for the r^{th} mode in the basis set and $\mathbf{X}_i$ is the $N \times 1$ applied displacement vector. The reactions forces, $\mathbf{F}_l$ are then extracted from the finite element model for the applied displacement field. The calculated reaction forces are then used as a static load terms in the NLROM equation of motion of the system as

$$\omega_r^2\mathbf{q}_r + \sum_{i=1}^{m}\sum_{j=1}^{m} \mathbf{B}_r(i,j)\mathbf{q}_i\mathbf{q}_j + \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} \mathbf{A}_r(i,j,k)\mathbf{q}_i\mathbf{q}_j\mathbf{q}_k = \phi_r^{\mathbf{T}}\mathbf{F}_l \tag{10}$$

where the nonlinear stiffness coefficients, A_r and B_r are solved using the method described in [8]. It is important to note that for the ED method the number of nonlinear static solutions required to find the stiffness coefficients for $m > 3$ [7] is

$$2\mathrm{m} + \frac{3\mathrm{m}!}{2(\mathrm{m}-2)!} + \frac{\mathrm{m}!}{6(\mathrm{m}-3)!} \tag{11}$$

2.2 Implicit Condensation and Expansion (ICE)

The implicit condensation and expansion method is similar to the enforced displacement method but instead of applying a set of displacements in the linear mode shapes of the system, a set of forces are applied that would each excite only one linear mode if the structure were linear. The general equation for a forcing with multiple modes can be given as

$$\mathbf{F}_l = \mathbf{M}(\phi_1\hat{\mathbf{f}}_1 + \phi_2\hat{\mathbf{f}}_2 + \ldots + \phi_m\hat{\mathbf{f}}_m) \tag{12}$$

where $\mathbf{F}_l$ is the vector of applied forces, $\mathbf{M}$ is the mass matrix and $\mathbf{f}_r$ is the scaling factor for the r^{th} modal force. The mass matrix is included above to exploit orthogonality, so that the shapes $\phi_r\hat{\mathbf{f}}_r$ each only excite one mode of the linear system [9]. For the ICE method the number of required load permutations required to generate the static response data [1] (for m >3) is

$$2\mathrm{m} + 2\frac{\mathrm{m}!}{(\mathrm{m}-2)!} + \frac{4\mathrm{m}!}{3(\mathrm{m}-3!)} \tag{13}$$

2.3 Nonlinear Normal Modes

Nonlinear Normal Modes are the foundation of evaluating the accuracy of the computed ROMs in this work. NNMs defined by Vakakis, Kerschen and others [5] as “a not necessarily synchronous periodic solution of the conservative, nonlinear equations of motion.” NNMs provide the backbone of the nonlinear dynamics of the system and provide strong quantitative metrics for which to evaluate the ROM [9]. Typically NNMs are plotted on a frequency versus energy plot, or FEP, which shows how the resonance frequency changes with energy of the system. The NNMs of the undamped ROMs as represented in Eq. (6) are computed in this work using a pseudo-arclength continuation method developed in [9]. The continuation algorithm uses the linear mode shape to initiate the branch for each NNM of the system. From there the algorithm finds a branch of solutions that satisfy the following shooting function.

$$\mathbf{H}(\mathbf{T},\mathbf{q}_O,\mathbf{q}_O) = \left\{ \begin{array}{c} \mathbf{q}(\mathbf{T},\mathbf{q}_O,\dot{\mathbf{q}}_O) \\ \dot{\mathbf{q}}(\mathbf{T},\mathbf{q}_O,\dot{\mathbf{q}}_O) \end{array} \right\} - \left\{ \begin{array}{c} \mathbf{q}_O \\ \dot{\mathbf{q}}_O \end{array} \right\} = \{\mathbf{0}\} \tag{14}$$

where T , $\mathbf{q}_0$, and $\dot{\mathbf{q}}_0$ are the period of integration, initial modal displacements, and initial modal velocities respectively.

2.4 Proposed Methodology for Creating a Valid ROM

Previous work [4] established a methodology for generating valid ROMs of a finite element model using the convergence of NNMs as an accuracy metric; this study follows the general procedure with some additions. Before generating the ROM the analyst must analyze the loading environment the structure will experience to determine the range for which the ROM must maintain accuracy. This includes both the frequency and amplitude of the loadings. Prior work [4] focused on finding a ROM that would accurately reproduce each NNM individually. This work seeks to generate a ROM that will accurately reproduce a set of NNMs to assure that the model is accurate over a range of frequency and energy.

In prior works, it has proven helpful to consider static load cases in terms of how much the nonlinear deformation deviates from a linear response. This is done by defining

$$\mathbf{x}_{rat} = \frac{max\left(\mathbf{w}_{NL}\right)}{max\left(\mathbf{w}_{linear}\right)} \tag{15}$$

where x_{rat} is the ratio of the calculated displacement of the nonlinear system to the displacement that would be computed using a linear model. This ratio is less than one for hardening nonlinearities, and can also be greater than one for structures that exhibit snap-through or other softening nonlinearities.

The general procedure for creation of valid ROM is:

1. Conduct a pre-processing step to determine the dominant modes and initial load scaling factors for each NNM branch
   (a) Define static loads that will generate $\mathbf{x}_{rat}$ values between 0.80 - 0.9 for hardening nonlinearity or $\mathbf{x}_{rat}$ values between 1.05 -1.20 for softening nonlinearity for each linear mode based on [10]. This provides a baseline for load scaling factors for each linear mode in the basis set.
   (b) Use $\mathbf{q} = \Phi^{\mathbf{T}}\mathbf{M}\mathbf{x}$ to compute the modal displacement of the each mode in response to the applied load in order to find the strongest modal coupling to mode of interest. This determines which modes should be included in the basis set of the ROM by selecting the modes in the order of dominance in the static response.
2. Generate a 1-mode ROM and compute the corresponding NNM
3. Add additional modes as found to be strongly coupled to the desired NNM branch in step (1b)
4. Recompute the NNM for the multi-mode ROM and compare the results with the previously computed NNM by checking:
   (a) Convergence of NNMs energy at a specific frequency.
   (b) Check the periodicity, ε , of the undamped free response of the full finite element model using the NNM solution transferred back to the real domain as initial condition. The periodicity of the response is evaluated using

   $$\frac{\|\mathbf{X}_T - \mathbf{X}_0\|}{\|\mathbf{X}_0\|} = \varepsilon \tag{16}$$

   where X_0 and X_T are the displacement vectors of system for the initial conditions and at the end of one period respectively, and $\|\ \|$is the norm operator on a vector. The lower the ε value the more accurate representation the ROM is of the full FEM. A value of ε below $10^{-1}$is desirable. This step should be used sparingly and as a final check due to the computational cost associated with integrating the full FEM over one period.
5. After evaluation of the above parameters if convergence has not been achieved, the analyst can either:
   (a) Add additional modes to the basis set in the order of their dominance by using one of the following methods:

i. Refer to the statically coupled modes found in the pre-processing presented in step (1b).

ii. Computing the modal amplitudes found in the free response of the full FEM model in step (4b). This is accomplished by transferring the real domain response to modal amplitudes as a function of time $\mathbf{q}(t) = \Phi^{\mathrm{T}}\mathbf{M}\mathbf{x}(t)$. This transformation gives insight into which modes are contributing to the dynamic response but may not be included in the ROM.

(b) Change load scaling factors of the basis vectors.

6. Repeat steps (4-5) until convergence is achieved for the current NNM.

7. Expand ROM to next NNM using steps (2-6) until the ROM is accurate over the desired range of frequency and energy encompassed by the NNMs.

3 Numerical Results of an Asymmetric Curved Beam

The methodology described above was tested on a curved beam using both the ED and ICE methods of generating ROMs. The beam was modeled using 80 3-D beam finite elements generated in Abaqus® with a total of 480 DOF constrained to in-plane motion and rotation. The curved beam has a rectangular cross section with clamped-clamped end conditions as shown in Fig. (1). The properties of the beam can be found in Table 1.

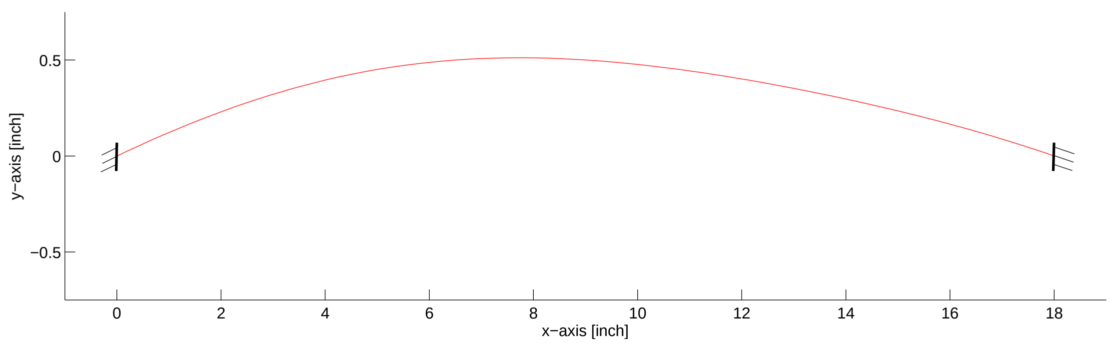

Figure 1: Curved beam diagram showing the curvature and asymmetry with clamped clamped boundary conditions

The curved beam has an 81.5 inch radius of curvature with a slight asymmetry imposed on it. Note that the figure of the beam is not to scale, this is in order to show the curvature of beam and the slight asymmetry present.

Table 1: Properties of Asymmetric Curved Beam

Young's Modulus $\left[\frac{lb}{in^2}\right]$	Poision's Ratio $[\nu]$	Density $\left[\frac{lb}{in^3}\right]$	Thickness $[in]$	Width $[in]$	Length $[in]$
10600000	0.325	0.000259	0.09	1	18

3.1 Pre-Processing

The first step in creating the ROMs is to apply static loads as described in step 1(a). The static coupling results from the pre-processing are found in Table 2. Notice that for the first four linear modes that the six most strongly coupled modes are all the same, this is helpful since, we seek to create one ROM that accurately reproduces the first four NNMs. The scaling factors are chosen such that they properly excite the nonlinearities of the structure. The original load factors were calculated so that the x_{rat} value was between 0.8 and 0.9 for hardening nonlinearities and 1.05 to 1.2 for softening nonlinearities as recommended by [7,11]. Load scaling factors were also perturb outside the recommended range to test the sensitivity of their values on generating ROMs for curved beams.

Table 2: Static modal coupling rankings for first four linear modes used to create ROMs basis set

Linear Mode	Frequency [Hz]	Contribution (highest $\rightarrow$ lowest)
1	145	[1, 3, 2, 4, 5, 7, 15, 22, 6, 38]
2	270	[2, 3, 1, 4, 5, 7, 22, 15, 9, 42]
3	415	[3, 1, 2, 4, 5, 7, 15, 6, 9, 22]
4	514	[4, 3, 1, 2, 5, 7, 22, 50, 9, 57]

3.2 Computed NNMs

Once the pre-processing is complete the analyst is now able to compute the NNMs and validate the ROM. The results for the first NNM using both the ED and ICE method with similar basis sets are shown in Fig. (2). The legend shows the method of ROM generation and the modes included within the basis set. The designation "1.1" represents a dual mode included within the basis set for the ED method corresponding to the 1st bending mode. The ROMs presented in Fig. (2) contained scaling factors that created x_{rat} values of approximately 0.85 based on the pre-processing.

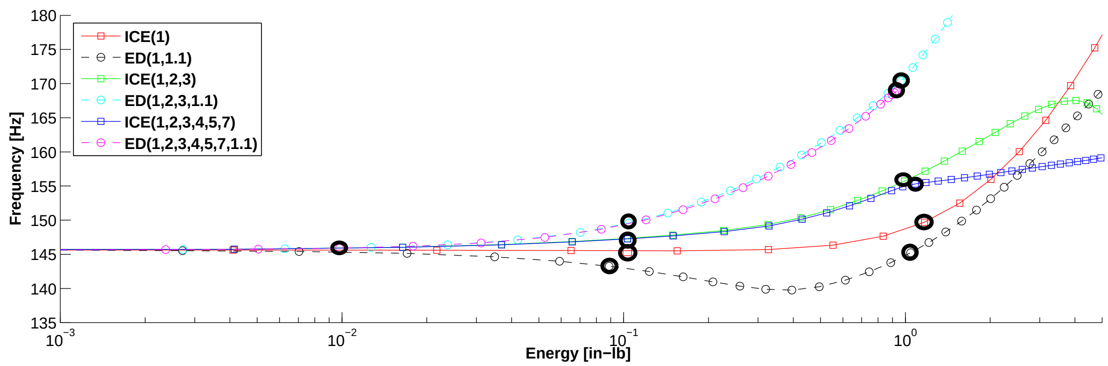

Figure 2: 1st NNM computed from ROMs using similar basis sets for both ED and ICE methods. The black circles represent NNM solution points used to test the periodicity of the full FEM model in Table 3.

In Fig. (2) it is shown that for both the ED and ICE method that the one-mode ROMs varies significantly from the multi-mode ROMs. As additional modes are included within the basis set the NNMs show less change in energy of the system for a specific period. Each method of model reduction shows to trend towards a different solution of the NNM. In order to validate which method is approaching the true solution the periodicity of the full order finite element model to initial conditions of the computed NNMs is evaluated. The initial modal conditions computed from the NNMs are transferred from the modal domain to the real and used as initial conditions in the free response of the system as described is step (4a). If the response shows to be periodic over one period of the solution then the computed NNM is a solution to the FEM. Fig. (3) shows the free response due to initial conditions taken from point 3 on the NNM plot near an Energy of 1 $[in-lb]$ for ICE ROMs for a set of nodes of the beam over time. The modal amplitudes of the linear modes within the dynamic response are plotted on the right side.

Table 3: Periodicity (ε) from Eq. (16) for the points shown with black circles in Fig. (2)

Point	Energy [in-lb]	ED(1)	ED(1, 2, 3)	ED(1, 2, 3, 4, 5, 7)	ICE(1)	ICE(1, 2, 3)	ICE(1, 2, 3, 4, 5, 7)
1	≈0.01	0.0303	0.00269	0.00193	0.023	0.00082	0.00020
2	≈0.10	0.132	0.0303	0.0235	0.129	0.012	0.00082
3	≈1.00	0.460	0.316	0.278	0.714	0.085	0.01967

The free response due to initial conditions taken from points on the NNM plot for ED ROMs for a set of nodes of the beam over time and the modal amplitudes of the linear modes within the response are presented in Fig. (4) on the left and right side respectively. These figures contain the results from several ROMs that were computed following the procedure explained earlier. Initially, a one-mode ICE ROM was computed and the periodicity was checked at a few points. When this revealed that the ROM was inadequate, additional modes were added according to the ordering in Table 2. While the resulting ROM was greatly improved, an additional ROM was created with the next most dominant modes to check convergence. The same approach was applied to ED method. These figures provide a strong qualitative metric to evaluate ROMs, Table 3 presents the periodicity,ε, of the free response solution, which provides a quantitative metric.

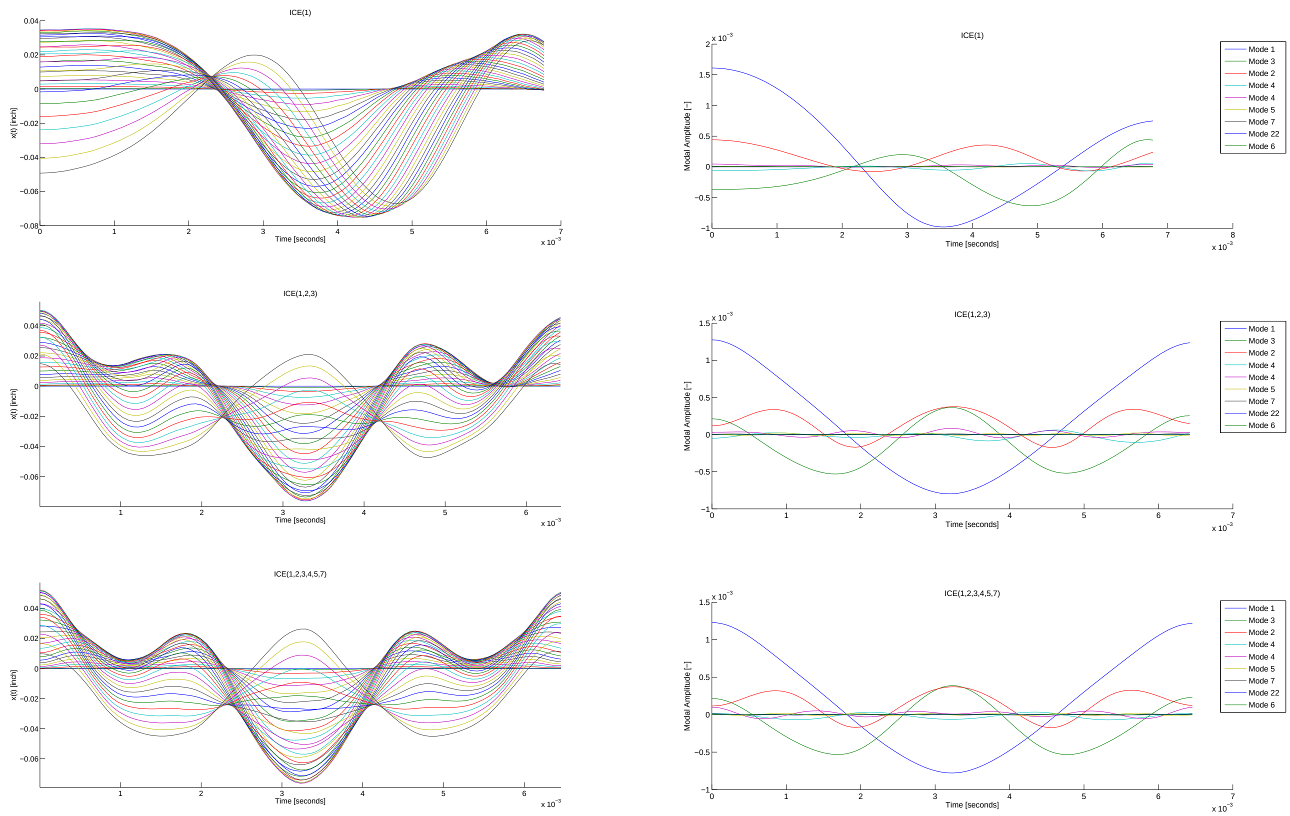

Figure 3: Left: Time histories found from integrating the initial conditions from NNM computed using ICE ROMs at points (3) near Energy = 1 $[in-lb]$ in Fig. (2); Right: Time histories of modal amplitudes of the linear modes found in the dynamic response of the system shown on the left

The results of Table 3 show that for each energy point the ICE ROM provided a more periodic solution of the FEM when compared to the ED results. It is also noted that there is significant increase in periodicity of the solution as more modes are added to the basis set as seen in Figs. (3,4). As energy of the system increases the periodicity of the ROMs decreases but the 3-mode and 6-mode ICE ROMs still maintain an ε value of less than $10^{-1}$for all three energy points.

In both Figs. (3,4) the one mode ROMs for ED and ICE have a dominant first mode that is included in the basis as shown in the plot but modes 3 and 2 are on the same order within the response so they should be included as well. The legend ranks the modal amplitude of the 9 most strongly coupled modes in the dynamic response. The modes that show to have the strongest influence on the response correspond to those presented in Table 2. This shows that the pre-processing was effective at predicting the statically coupled modes that will show up in the dynamic response of the curved beam.

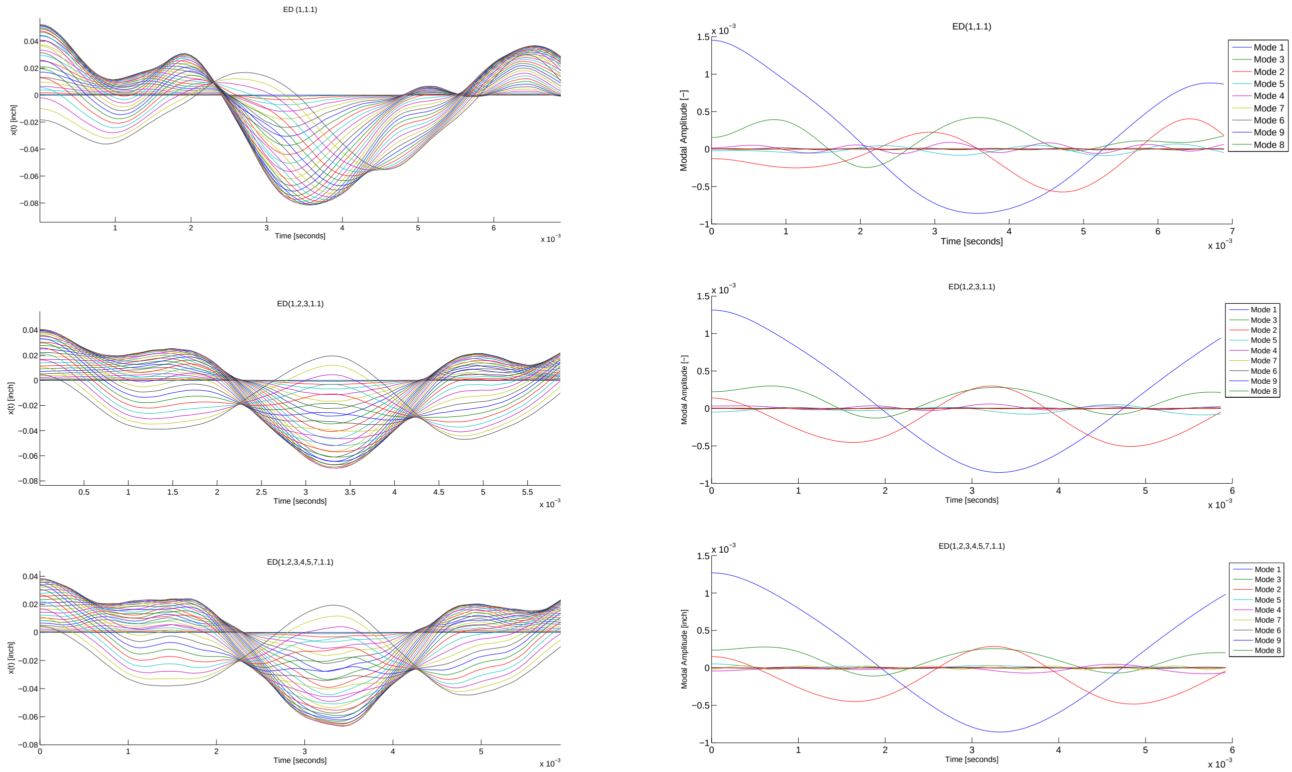

Figure 4: Left: Time histories found from integrating the initial conditions from NNM computed using ED ROMs at points (3) near Energy = 1 $[in - lb]$ in Fig. (2); Right: Time histories of modal amplitudes of the linear modes found in the dynamic response of the system shown on the left

The ED method was tested with additional dual modes within the basis set but the results were still less accurate than the ICE method ROMs containing a lower number of modes. Since the ICE method showed a more accurate computation of the 1st NNM than the ED method for a smaller set of modes, this work focused on expanding the ROM of just the ICE method to higher NNMs. An important aspect to note between the two methods is that as more modes were included to the basis set the ICE method encountered an internal resonance branch at 157 Hz as seen in Fig. (2). The continuation algorithm was diverted down this branch and then encountered a region where the step size became very small and hence never returned to the main branch. In contrast, the ED ROMs didn't include this internal resonance so the algorithm readily computed the branch up to a frequency shift of 250 Hz. Hence, if the backbones are of primary interest then the ED ROM might be preferred for this system.

Once it was discovered which ROMs accurately capture the 1st NNM, it was sought to test the ROM for the 2nd NNM. Since the 2nd NNM was found to be coupled to the same modes that were important for NNM1, it was not necessary to create any new ROMs to investigate the convergence of NNM2. The plot of the second NNM is shown in Fig. (5) where the converged ROMs for the ICE method are shown as well as low order ROMs to show that they lead to the converged solutions of each.

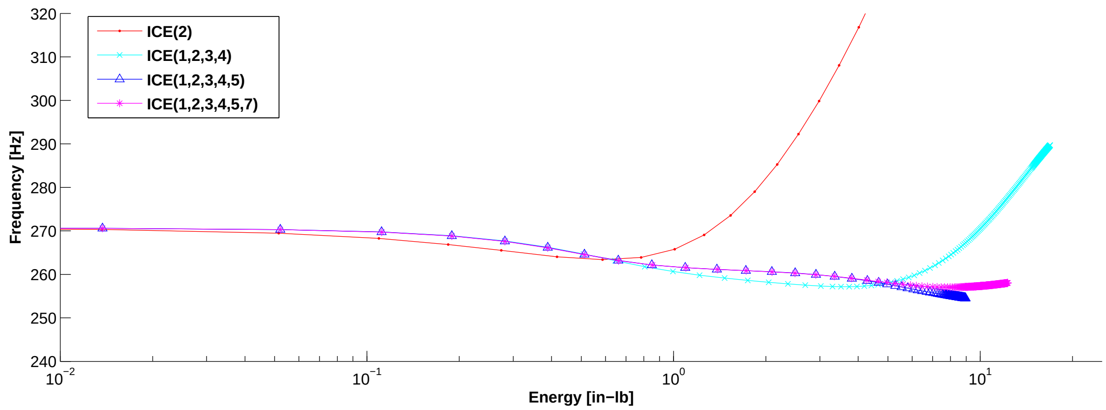

Figure 5: 2nd NNM computed from the ROMs using: 1-mode ROM at the 2nd NNM linear mode branch; 4-, 5-, 6-mode ICE ROM from the 1st NNM computed

Fig. (5) shows that the one-mode ROM of the 2nd NNM varies significantly from the multi-mode ROMs. There is a change between the four-mode and five-mode ROMs solution of the NNM but between the 5-mode and 6-mode ROMs the solution remains consistent. The 6-mode ICE ROM showed convergence for the 2nd NNM and was used to calculate the 3rd and 4th NNMs. The NNMs calculated from a one-mode ROM, the original 6-mode ROM and higher order ROMs are presented in Fig. (6) for the 3rd NNM. The addition of the 7-, 8-, and 9-mode ROMs shown in Fig. (6) were added to determine if the solution of the NNM for the 6-mode ROM would remain in the same region. It is shown that the solution did remain converged when adding the higher mode ROMs.

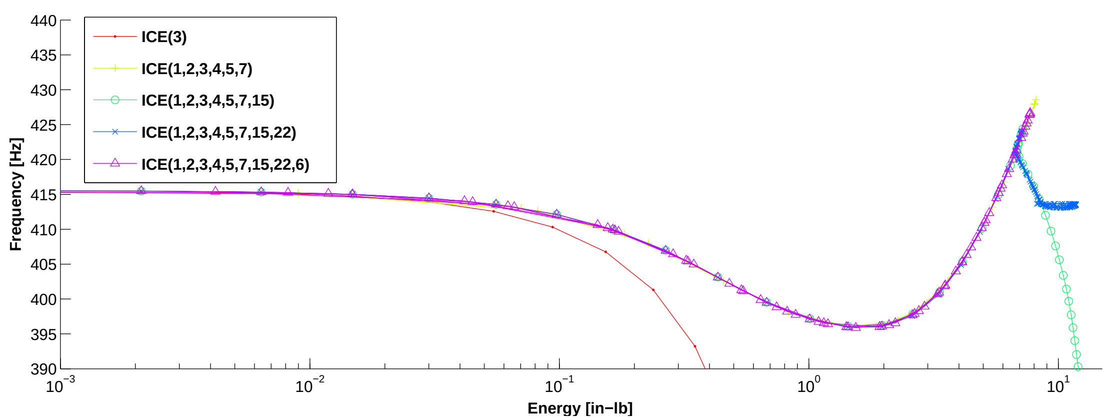

Figure 6: 3rd NNM computed from the ROMs using: 1-mode ICE ROM at the 3rd NNM linear mode branch; 6-mode ICE ROM from the 1st NNM computed; 7-, 8-, 9-mode ICE ROMs

The 4th NNM computed is shown in Fig. (7) with a 1-mode ROM, the original 6-mode ROM and additional higher order ROMs. As with the 3rd NNM, additional higher order ROMs are included in order to test if the NNM solution remains in the same region. The solution of the NNMs remained converged until the ROMs encountered internal resonances as more modes were added to the basis set. The 4th NNM presented in Fig. (7) shows the effect on the computed NNM of the ICE method when adding linear modes to the basis set. As more modes are added to the basis set the NNM computed encounters internal resonances at lower frequencies.

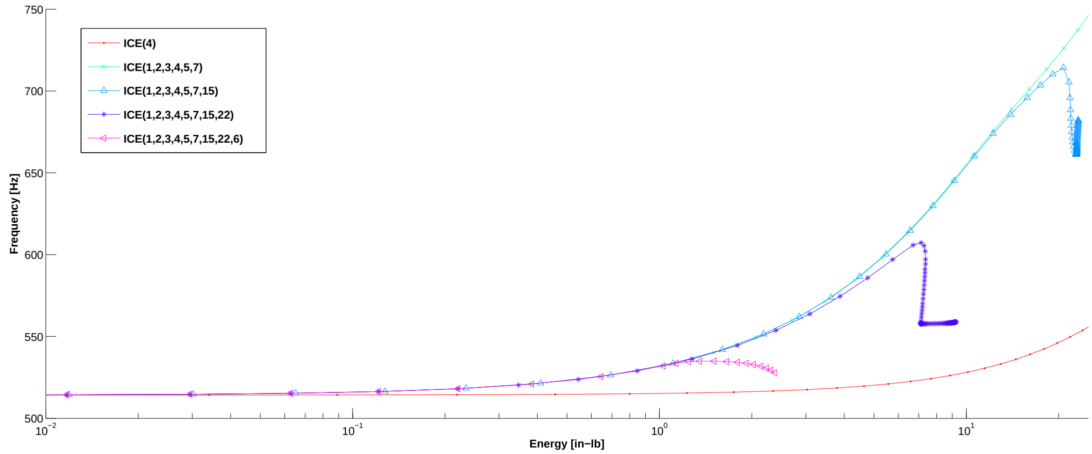

Figure 7: 4th NNM computed from the ROMs using: 1-mode ICE ROM at the 4th NNM linear mode branch; 6-mode ICE ROM from the 1st NNM computed; 7-, 8-, 9-mode ICE ROMs

Table 4 shows the periodicity values, ε, for the 6-mode ROM for each computed NNM. The table shows that the original 6-mode ROM stays consistently accurate up to that energy level of loading based on the periodicity of the full FEM model from computed NNM solutions. Since the ROM is accurate up to an energy level of 1 $[in - lb]$ and for the first 4 NNMs ranging from 147 Hz to 750 Hz then if the loading environment is within those bounds the ROM can be considered an acceptable approximation of the full FEM model.

Table 4: Periodicity (ε) from Eq. (16) for each NNM taken at solutions near an Energy = 1 $[in - lb]$ for: 1-mode ICE ROM at the NNM linear mode branch; 6-mode ICE ROM with scale factors of (0.45, 0.21, 0.22, 0.31, 0.21, 0.20)

NNM	1-mode ICE ROM (ε)	ICE(1, 2, 3, 4, 5, 7) (ε)
NNM1	0.714	0.0197
NNM2	0.402	0.0122
NNM3	1.850	0.0051
NNM4	0.335	0.0153

Note that this study used energy of the computed NNM as one of the metrics to evaluate the range that the ROM must maintain accuracy. Other metrics could be used including maximum displacement at a specific point along the beam or force values required to hold a steady state response at a NNM solution. This paper focused on energy due to the common representation of NNMs on a FEP plot.

3.3 Load Scaling Sensitivity Analysis

The second aspect of this work was to test the sensitivity of the ROMs to the load case factors used to extract the non-linear stiffness term. In previous work [7,10] it was found that an $\mathbf{x}_{rat}$ value of 0.8 and 0.9 was desired to properly excite the nonlinearity of a structure showing hardening while an $\mathbf{x}_{rat}$ value of 1.05 and 1.2 was desired for structures with softening nonlinearities. The load scaling factors were altered outside the bounds of those that generated $\mathbf{x}_{rat}$ values between 0.8 and 0.9 in order to test how they affect the computed NNMs. Table 5 shows the load scaling factors used that are associated with each $\mathbf{x}_{rat}$ value for the 1-mode ROMs that are presented in Fig. (8). The increase in the load scaling factor results in a lower $\mathbf{x}_{rat}$ value. The system energy changes slightly as the load scaling factor is increased until it is increased to a value of 1.86 causing an $\mathbf{x}_{rat}$ of 0.56 where the energy and period of the NNM solution significantly changes.

Table 5: Table of load scaling factors for one-mode ROM using ICE method with associated $\mathbf{x}_{rat}$ values

ROM	Load Scaling Value	$\mathbf{x}_{rat}$
1	0.28	0.89
2	0.45	0.85
3	0.62	0.80
4	0.78	0.76
5	0.93	0.72
6	1.86	0.56

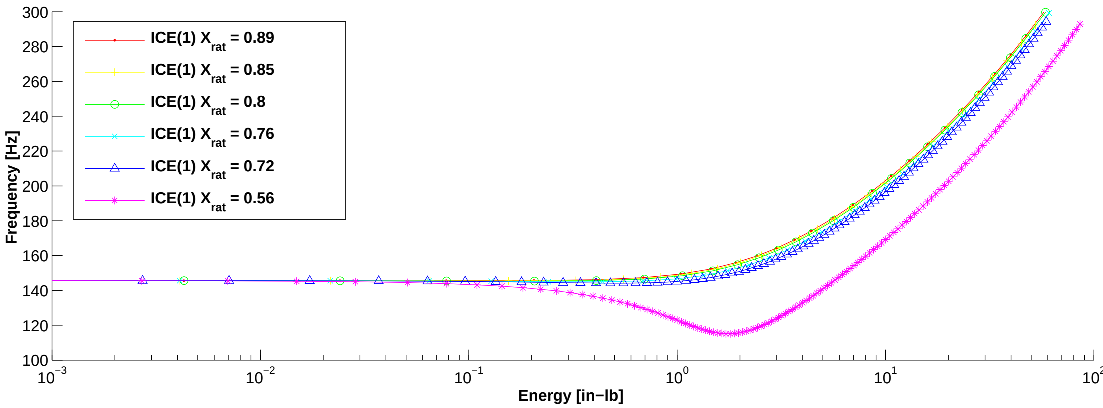

Figure 8: NNM1 computed from ICE(1) ROM using x_{rat} values from 0.89 to 0.55

A 6-mode ROM using the ICE method was also analyzed to evaluate the sensitivity of the ROMs with a larger basis set to load case factors. The method for determining load scaling factors for a multiple mode ROM is less straightforward than a single mode ROM. The original scaling values were determined for each linear mode individually to determine which values would generate $\mathbf{x}_{rat}$ values at 0.9, 0.85 and 0.8. These values were then perturbed similar to the single-mode ROMs, the load scaling values and calculated $\mathbf{x}_{rat}$ values for the 6-mode ROM are presented in Table 6. The x_{rat} values presented in Table 6 are the average of the three most dominant modes x_{rat} values.

Table 6: Table of load scaling factors for 6-mode ROM using ICE method with associated $\mathbf{x}_{rat}$ and ε values

ROM	Load Scaling Values	$\mathbf{x}_{rat}$ values	Periodicity (ε) at E = 1 [in-lb]
1	[0.28 0.12 0.13 0.21 0.13 0.10]	0.90	0.0305
2	[0.45 0.21 0.22 0.31 0.21 0.20]	0.85	0.0197
3	[0.62 0.30 0.30 0.41 0.28 0.17]	0.80	0.0113
4	[1.86 0.90 0.90 1.23 0.84 0.41]	0.60	0.0097
5	[6.82 3.30 3.30 4.51 3.08 2.97]	0.34	0.0117
6	[13.02 6.30 6.30 8.61 5.88 5.67]	0.25	0.0681
7	[25.42 12.30 12.30 16.81 11.48 11.07]	0.18	0.823

The load scaling values were extended to values 90 times the scaling that would provide and $\mathbf{x}_{rat}$ value of 0.8 for each individual mode. The NNMs of the ROMs computed using the load scaling factors presented in Table 5 are shown in Fig. (9). In Table 5 the periodicity of the 6-mode ROM remains consistent up to an x_{rat} value of 0.25 then greatly changes at a x_{rat} value of 0.18. This corresponds to the results seen in Fig. (9) where the NNMs of the computed ROMs show to remain in the same region from $\mathbf{x}_{rat}$ values of 0.9 to 0.34 corresponding to a 23 amplification in load scaling factors. Once the scaling factor was amplified to 45 times the base value a noticeable amount of change of the energy of the NNM solution can be seen. When approaching an increase of 90 times a significant amount of change of energy of the system can be noticed as the NNM starts to diverge from the other NNMs. Note that the ε value reached an optimum value of 0.0097 at an x_{rat} value of 0.60 for the curved been using the same basis vector shown in Table 6.

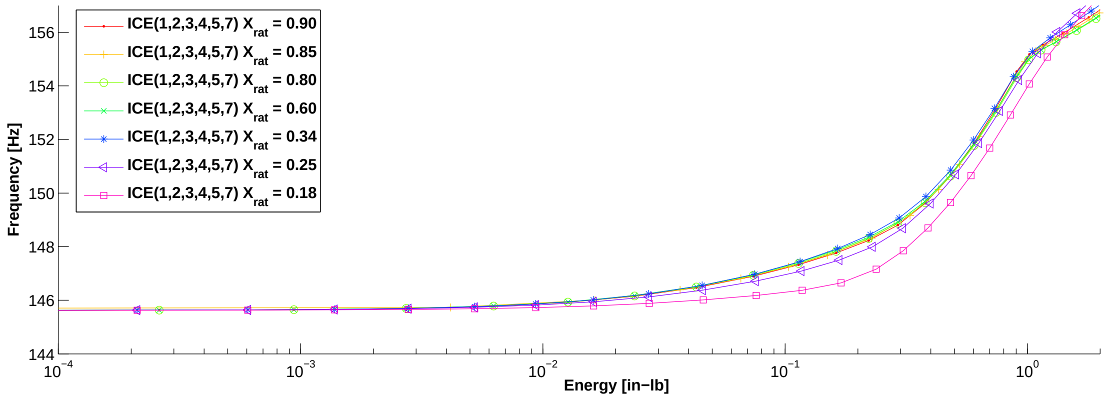

Figure 9: 1st NNM Computed from ICE(1,2,3,4,5,7) ROM various x_{rat} values from 0.90 to 0.18

4 Conclusion

This work implemented the procedure put forth in [4] on a structure with curvature whose nonlinear dynamics are highly complex. In addition this work expanded on [4] by validating the ROM for multiple NNMs as required by the loading amplitude and frequency range of the structure. This addition provides an approach useful to an analyst who will need to ensure that only one ROM is accurate for the model. A study was conducted on the sensitivity of load scaling values for the ICE method when generating a ROM. This is helpful for an analyst in order to determine whether to add another mode to the ROM, which comes with a substantial cost, or alter load scaling factors for the modes within the basis set.

The ROM creation procedure was applied to a curved beam with geometric nonlinearity and fixed boundary conditions at both ends. It is shown that the methodology proposed by [4] provides a strong foundation for generating and evaluating ROMs for the curved beam. The accuracy of the ROM for both methods was significantly increased as additional modes were included in the basis set. The ED and ICE methods converged towards their own set of approximation of the NNM. In order to evaluate which method was approaching the correct solution of the NNM, the periodicity of the full FEM to initial conditions of the NNM solution was compared. The ICE method of generating a ROM provided a more accurate representation of the full FEM than the ED method for the first NNM using a smaller basis set of modes. Further investigation needs to be done on determining why the ED method approached a significantly different approximation of the NNM.

The work then expanded the 6-mode ICE method ROM computed for the 1st NNM to higher NNMs. Due to the strong modal coupling of the curved beam the procedure of expanding ROMs to higher NNMs as needed by the loading environment scaled well. No additional modes were required to be added to basis set of the original 6-mode ICE ROM in order to remain accurate, up to the specified energy level, for the first four NNMs. This was evaluated by testing the periodicity of the system at similar energy points of the NNM. This ensures that the ROM is accurate for the amplitude and frequency of the loading environment represented in this work as energy and frequencies of the computed NNMs.

The ICE method for curved structures showed to be less sensitive to the scaling factors for ROMs that have larger basis sets. This was shown by the difference in the computed NNMs of the 1-mode and 6-mode ROMs for a range of load scaling factors. The 1-mode ROM showed a consistent change in NNM solution as the load scaling factors were perturbed. The 6-mode ROM remained consistent for a large window of deviation of load scaling factors. The periodicity of the full FEM to the NNM solution of the 6-mode ROM corresponded with the NNMs on the FEP plot for the range of load scaling factors applied.

Overall the procedure established in [4] provided a straightforward method for generating an accurate ROM of a curved beam for a single NNM. The procedure then scaled well to higher NNMs as required due to the strong modal coupling of the beam. The effect of a structure's curvature on computed NNMs and ROM generation needs to be evaluated in further detail in order to provide a stronger metric off determining load scaling values. It is also recommended that this methodology be tested on a more complicated structure in order to further validate the process.

Acknowledgements

The authors also acknowledge Joseph Hollkamp from the Air Force Research Laboratory's Structural Sciences Center, for providing the Abaqus interface that was used in this work.

References

1. R. W. Gordon and J. J. Hollkamp, "Reduced-Order Models for Acoustic Response Prediction," Air Force Research Laboratory (AFRL), Wright-Patterson Air Force Base, OH AFRL-RB-WP-TR-2011-3040, 2011.
2. G. Tzong and S. L. Liguore, "Verification Studies on Hypersonic Structure Thermal/Acoustic Response and Life Prediction Methods," presented at the 54th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, 2013.
3. R. Perez, X. Q. Wang, and M. P. Mignolet, "Nonintrusive structural dynamic reduced order modeling for large deformations: Enhancements for complex structures," Journal of Computational and Nonlinear Dynamics, vol. 9, 2014
4. Kuether, R., & Allen, M. "Validation of Nonlinear Reduced Order Models with Time Integration Targeted at Nonlinear Normal Modes". Nonlinear Dynamics, Volume 1 Conference Proceedings of the Society for Experimental Mechanics Series, 363-375.
5. A. F. Vakakis, "Non-linear normal modes (NNMs) and their applications in vibration theory: an overview," Mechanical Systems and Signal Processing, vol. 11, pp. 3-22, 1997.
6. J. J. Hollkamp, R. W. Gordon, and S. M. Spottswood, "Nonlinear modal models for sonic fatigue response prediction: a comparison of methods," Journal of Sound and Vibration, vol. 284, pp. 1145-63, 2005.
7. M. P. Mignolet, A. Przekop, S. A. Rizzi, and S. M. Spottswood, "A review of indirect/non-intrusive reduced order modeling of nonlinear geometric structures," Journal of Sound and Vibration, vol. 332, pp. 2437-2460, 2013.
8. A. A. Muravyov and S. A. Rizzi, "Determination of nonlinear stiffness with application to random vibration of geometrically nonlinear structures," Computers & Structures, vol. 81, pp. 1513-1523, 2003.
9. Kuether, R., Deaner, B., Hollkamp, J., & Allen, M. "Evaluation of Geometrically Nonlinear Reduced-Order Models with Nonlinear Normal Modes". AIAA Journal, 1-13.
10. J. J. Hollkamp, R. W. Gordon, and S. M. Spottswood, "Nonlinear sonic fatigue response prediction from finite element modal models: a comparison with experiments," in Proceedings of the 44th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, 2003.
11. R. W. Gordon and J. J. Hollkamp, "Reduced-Order Modeling of the Random Response of Curved Beams using Implicit Condensation," in 47th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, Newport, Rhode Island, 2006.