Derivation, Validation and Comparison of the NIFO and NIXO Algorithms for SDOF Systems

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Contents

Abstract

This report presents a detailed derivation of a new system identification approaches called NIXO methods, from Nonlinear Identification through eXtended Outputs. The new algorithms are very similar to a popular family of nonlinear estimators called NIFO methods (from Nonlinear Identification through Feedback of the Outputs). These methods estimate the underlying linear Frequency Response Function (FRF) as well as the parameters describing the mechanical system's nonlinearities. However, the nonlinear parameters returned by the NIFO algorithms are complex and usually vary with frequency, while the NIXO estimators find them as real and frequency-independent. In this work the methods are evaluated numerically using a single degree of freedom system with a cubic nonlinearity. Results obtained using NIFO and NIXO are presented and compared to each other.

Keywords: Nonlinear System Identification, NIFO methods, NIXO methods, Nonlinear H-Estimators

1 Derivation of Nonlinear System Identification Methods

In this section a theory behind three different types of nonlinear system identification algorithms is presented. These three algorithms are:

– $\mathbf{H}_1$– and $\mathbf{H}_2$–based NIFO methods (a.k.a. modified $\mathbf{H}_1$ and $\mathbf{H}_2$ algorithms),

– $\mathbf{H}_1$ and $\mathbf{H}_2$ Nonlinear Identification through eXtended Outputs and

– $\mathbf{H}_1$ and $\mathbf{H}_2$ Nonlinear Identification through eXtended Outputs with Linear Data Provided.

Derivations start with steps common for all the algorithms, then they fork and focus on each method separately. For simplicity, we would like to consider a single degree of freedom (SDOF) mechanical system described with equations of motion (EOM) defined in Eqs. (1) or (2). In future work the algorithms will be extended to multi-DOF systems.

Let us consider a mechanical system described by one of the following EOMs:

$$m\ddot{x} + c\dot{x} + kx + c_2\dot{x}|\dot{x}| + c_3\dot{x}|\dot{x}|^2 + \cdots + k_2 x|x| + k_3 x|x|^2 + \cdots = f(t), \tag{1}$$

$$m\ddot{x} + c\dot{x} + kx + c_2\dot{x}^2 + c_3\dot{x}^3 + \cdots + k_2 x^2 + k_3 x^3 + \cdots = f(t), \tag{2}$$

where $m,\ c,\ k,\ c_i$ and k_j $(i,\ j \in \mathbb{N})$ are *real* and *constant* parameters, $x(t)$ is the response of the system excited for certain initial conditions with a forcing function $f(t)$. If the individual time functions are expressed as in (3), then Eqs. (1) or (2) become equivalent to Eq. (4) with $D(\Omega) = k - m\Omega^2 + ic\Omega$ and *frequency–independent* c_j's and k_j's.

$$\begin{aligned} x(t) &= \mathrm{Re}\{X\ e^{i\Omega t}\} & f(t) &= \mathrm{Re}\{F\ e^{i\Omega t}\} \\ \dot{x}|\dot{x}|^{j-1}(t) &= \mathrm{Re}\{dY_j\ e^{i\Omega t}\} = \dot{x}^j(t) & x|x|^{j-1}(t) &= \mathrm{Re}\{Y_j\ e^{i\Omega t}\} = x^j(t) \end{aligned} \tag{3}$$

$$D(\Omega)\ X(\Omega) + c_2\ dY_2(\Omega) + c_3\ dY_3(\Omega) + \cdots + k_2\ Y_2(\Omega) + k_3\ Y_3(\Omega) + \cdots = F(\Omega), \tag{4}$$

Using N_{avg} spectral averages (obtained using e.g. a Hanning window) of signals $x(t)$, $f(t)$ and higher powers of $x(t)$ and $\dot{x}(t)$, Eq. (4) can be extended to form shown in Eq. (5). For SDOF systems, matrices $\mathbf{X}$, $d\mathbf{Y}_j$, $\mathbf{Y}_j$ and $\mathbf{F}$ have size of $1 \times N_{avg} \times m$, where m is a number of frequency samples.

$$\begin{aligned} D(\Omega)\ [X_1,\ \ldots,\ X_{N_{avg}}] + c_2\ [dY_{2,1},\ \ldots,\ dY_{2,N_{avg}}] + c_3\ [dY_{3,1},\ \ldots,\ dY_{3,N_{avg}}] + \cdots + \\ + k_2\ [Y_{2,1},\ \ldots,\ Y_{2,N_{avg}}] + k_3\ [Y_{3,1},\ \ldots,\ Y_{3,N_{avg}}] + \cdots = [F_1,\ \ldots,\ F_{N_{avg}}] \end{aligned} \tag{5}$$

$$D(\Omega)\ \mathbf{X} + c_2\ d\mathbf{Y}_2 + c_3\ d\mathbf{Y}_3 + \cdots + k_2\ \mathbf{Y}_2 + k_3\ \mathbf{Y}_3 + \cdots = \mathbf{F} \tag{6}$$

1.1 NIFO Algorithms

1.1.1 $\mathbf{H}_1$–based NIFO method (modified $\mathbf{H}_1$ algorithm)

Original NIFO estimator was first proposed in [1]. It can be obtained via bringing Eq. (6) to form presented in Eq. (7), where quantity H is a Frequency Response Function ($H(\Omega) = \frac{1}{D(\Omega)}$). The modified $\mathbf{H}_1$ algorithm is based on Eq. (9), which is obtained by right-multiplying Eq. (7) by matrix $\boldsymbol{\Delta}$ defined in Eq. (8). Note that Eq. (9) is valid for every individual frequency rate.

$$\mathbf{X} = \begin{bmatrix} H & c_2H & c_3H & \dots & k_2H & k_3H & \dots \end{bmatrix} \begin{bmatrix} \mathbf{F} \\ -d\mathbf{Y}_2 \\ -d\mathbf{Y}_3 \\ \vdots \\ -\mathbf{Y}_2 \\ -\mathbf{Y}_3 \\ \vdots \end{bmatrix} \Bigg| \boldsymbol{\Delta} \tag{7}$$

$$\boldsymbol{\Delta} = \begin{bmatrix} \mathbf{F}^H & -d\mathbf{Y}_2^H & -d\mathbf{Y}_3^H & \dots & -\mathbf{Y}_2^H & -\mathbf{Y}_3^H & \dots \end{bmatrix} \tag{8}$$

$$\begin{aligned} &\overbrace{\begin{bmatrix} S_{XF} & -S_{XdY_2} & -S_{XdY_3} & \dots & -S_{XdY_2} & -S_{XdY_3} & \dots \end{bmatrix}}^{\mathbf{b}} = \overbrace{\begin{bmatrix} H & c_2H & c_3H & \dots & k_2H & k_3H & \dots \end{bmatrix}}^{\hat{\mathbf{x}}} \bullet \\ &\qquad \bullet \underbrace{\begin{bmatrix} S_{FF} & -S_{FdY_2} & -S_{FdY_3} & \dots & -S_{FY_2} & -S_{FY_3} & \dots \\ & S_{dY_2dY_2} & S_{dY_2dY_3} & \dots & S_{dY_2Y_2} & S_{dY_2Y_3} & \dots \\ & & S_{dY_3dY_3} & \dots & S_{dY_3Y_2} & S_{dY_3Y_3} & \dots \\ & & & \ddots & & & \ddots \\ & \text{Hermitian Matrix} & & & S_{Y_2Y_2} & S_{Y_2Y_3} & \dots \\ & & & & & S_{Y_3Y_3} & \dots \\ & & & & & & \ddots \end{bmatrix}}_{\mathbf{A}} \end{aligned} \tag{9}$$

The $\mathbf{H}_1$–based NIFO algorithm results in obtaining multiple systems of linear equations of form $\mathbf{b} = \hat{\mathbf{x}}\mathbf{A}$ (each system corresponds to a different frequency sample). Matrix $\mathbf{A}$ is squared and for problems of our interest it is usually non-singular, thus it might be possible to accurately estimate the FRF $H(\Omega)$ and parameters c_j and k_j through solving Eq. (9).

Note that the nonlinear parameters c_j and k_j were introduced in Eqs. (1) or (2) as *real* and *constant* numbers. However, they are computed as *complex* and possibly *frequency-dependent*, since some of the parameters in Eq. (9) are complex and (as already mentioned above) the system of equations is solved for each frequency sample separately.

1.1.2 $\mathbf{H}_2$–based NIFO method (modified $\mathbf{H}_2$ algorithm)

The $\mathbf{H}_2$–based NIFO was first presented in [2]. Its main concept lies in extending Eq. (7) by additional outputs corresponding to the nonlinear EOM form, see Eq. (10). The modified $\mathbf{H}_2$ algorithm is based on the general linear model (GLM) form presented in Eq. (12). This model can be obtained through right-multiplying Eq. (10) by matrix $\boldsymbol{\square}$ defined in Eq. (11).

$$
\begin{bmatrix} \mathbf{X} \\ d\mathbf{X}_2 \\ d\mathbf{X}_3 \\ \vdots \\ \mathbf{X}_2 \\ \mathbf{X}_3 \\ \vdots \end{bmatrix}
=
\begin{bmatrix}
H & c_2H & c_3H & \dots & k_2H & k_3H & \dots \\
 & -1 & & & & & \\
 & & -1 & & & & \\
 & & & \ddots & & & \\
 & & & & -1 & & \\
 & & & & & -1 & \\
 & & & & & & \ddots
\end{bmatrix}
\begin{bmatrix} \mathbf{F} \\ -d\mathbf{Y}_2 \\ -d\mathbf{Y}_3 \\ \vdots \\ -\mathbf{Y}_2 \\ -\mathbf{Y}_3 \\ \vdots \end{bmatrix}
\qquad \Bigg| \, \square
\tag{10}
$$

$$
\square = \begin{bmatrix} \mathbf{X}^H & d\mathbf{X}_2^H & d\mathbf{X}_3^H & \dots & \mathbf{X}_2^H & \mathbf{X}_3^H & \dots \end{bmatrix}
\tag{11}
$$

$$
\begin{aligned}
&\overbrace{\begin{bmatrix}
S_{XX} & S_{XdX_2} & S_{XdX_3} & \dots & S_{XX_2} & S_{XX_3} & \dots \\
 & S_{dX_2dX_2} & S_{dX_2dX_3} & \dots & S_{dX_2X_2} & S_{dX_2X_3} & \dots \\
 & & S_{dX_3dX_3} & \dots & S_{dX_3X_2} & S_{dX_3X_3} & \dots \\
 & \textbf{Hermitian Matrix} & & \ddots & & & \ddots \\
 & & & & S_{X_2X_2} & S_{X_2X_3} & \dots \\
 & & & & & S_{X_3X_3} & \dots \\
 & & & & & & \ddots
\end{bmatrix}}^{\mathbf{U}}
=
\overbrace{\begin{bmatrix}
H & c_2H & c_3H & \dots & k_2H & k_3H & \dots \\
 & -1 & & & & & \\
 & & -1 & & & & \\
 & & & \ddots & & & \\
 & & & & -1 & & \\
 & & & & & -1 & \\
 & & & & & & \ddots
\end{bmatrix}}^{\hat{\mathbf{X}}} \bullet \\
&\bullet \underbrace{\begin{bmatrix}
S_{FX} & S_{FdX_2} & S_{FdX_3} & \dots & S_{FX_2} & S_{FX_3} & \dots \\
-S_{dY_2X} & -S_{dY_2dX_2} & -S_{dY_2dX_3} & \dots & -S_{dY_2X_2} & -S_{dY_2X_3} & \dots \\
-S_{dY_3X} & & -S_{dY_3dX_3} & \dots & -S_{dY_3X_2} & -S_{dY_3X_3} & \dots \\
\vdots & & \textbf{Hermitian Sub-matrix} & \ddots & & & \ddots \\
-S_{Y_2X} & & & & -S_{Y_2X_2} & -S_{Y_2X_3} & \dots \\
-S_{Y_3X} & & & & & -S_{Y_3X_3} & \dots \\
\vdots & & & & & & \ddots
\end{bmatrix}}_{\mathbf{B}}
\end{aligned}
\tag{12}
$$

The modified $\mathbf{H}_2$ algorithm results in obtaining multiple general linear models of form $\mathbf{U} = \hat{\mathbf{X}}\mathbf{B}$. Each of those models corresponds to a different frequency sample, just as in its twin algorithm derived in section 1.1.1. Matrix $\mathbf{B}$ is squared and for problems of our interest it is usually non-singular, thus the accurate estimation of the frequency response function $H(\Omega)$ and parameters c_j and k_j describing the nonlinearity might be possible through solving Eq. (12).

Note that the nonlinear parameters c_j and k_j were introduced in Eqs. (1) or (2) as *real* and *constant* numbers. However, they will be estimated as *complex* and possibly *frequency-dependent*, since some of the parameters in Eq. (12) are complex and (as already mentioned above) the system of equations is solved for each frequency sample separately.

Finally it is worth noting that rows of matrices $\mathbf{B}$ and $\mathbf{U}$ (ranging from the 2^{nd} to the last) are almost exactly the same. The rows of matrix $\mathbf{B}$ are just rows of $\mathbf{U}$ multiplied by -1. This simple observation saves time (and space) spent on the algorithm implementation. Using MATLAB notation we could write:

```
B(2:end, :)  = -U(2:end, :);
```

1.2 Nonlinear Identification through eXtended Outputs Algorithms

1.2.1 $\mathbf{H}_1$-NIXO

Straight-forward application of $\mathbf{H}_1$ estimator to a nonlinear system results in obtaining underdetermined system of equations, which estimated solution is not guaranteed to be accurate. In order to prove it, right-multiply Eq. (6) by $\mathbf{F}^H$ to obtain Eq. (13). It is valid for every individual frequency sample, thus it is possible to express each of these equations in a matrix form shown in Eq. (14). Note that the frequency sample number was indicated in the parameters sub- or superscripts, e.g. $S_{XF}(\Omega_i) = S^{\,i}_{XF}$ or $D(\Omega_i) = D_i$.

$$D(\Omega)\ \mathbf{X}\mathbf{F}^H + c_2\ d\mathbf{Y}_2\mathbf{F}^H + c_3\ d\mathbf{Y}_3\mathbf{F}^H + \cdots + k_2\ \mathbf{Y}_2\mathbf{F}^H + k_3\ \mathbf{Y}_3\mathbf{F}^H + \cdots = \mathbf{F}\mathbf{F}^H$$

$$D(\Omega)\ S_{XF} + c_2\ S_{dY_2F} + c_3\ S_{dY_3F} + \cdots + k_2\ S_{Y_2F} + k_3\ S_{Y_3F} + \cdots = S_{FF} \tag{13}$$

$$\begin{bmatrix} S^{\,1}_{XF} & & & S^{\,1}_{dY_2F} & S^{\,1}_{dY_2F} & & S^{\,1}_{Y_2F} & S^{\,1}_{Y_3F} & \\ & \ddots & & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \\ & & S^{\,m}_{XF} & S^{\,m}_{dY_2F} & S^{\,m}_{dY_2F} & & S^{\,1}_{Y_2F} & S^{\,1}_{Y_3F} & \end{bmatrix} \begin{bmatrix} D_1 \\ \vdots \\ D_m \\ c_2 \\ c_3 \\ \vdots \\ k_2 \\ k_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} S^{\,1}_{FF} \\ \vdots \\ S^{\,m}_{FF} \end{bmatrix} \tag{14}$$

As mentioned at the beginning of this section, Eq. (14) shows that straight-forward application of $\mathbf{H}_1$ estimator to a nonlinear system results in obtaining more unknowns than equations. To be more precise we obtained **2m** equations and $\mathbf{2m} + \mathbf{p_{damp}} + \mathbf{p_{stiff}}$ unknowns, since some of the parameters in Eq. (14) are complex numbers in general; p_{damp} and p_{stiff} are naturally numbers of nonlinear damping and stiffness terms in the equation of motion (1) or (2).

The solution to underdetermined system of equations is not unique, which can lead to inaccuracies in parameters' estimation. This issue might be overcome by providing input and output data sets collected in vibration tests where system oscillates at multiple different amplitudes. Such data can be used in populating number of equations in Eq. (14) while keeping the number of unknowns fixed. This is the main concept behind the new ***H***$_1$ *Nonlinear Identification through eXtended Outputs Estimator* ($\mathbf{H}_1$-NIXO). The the idea originates in two observations:

1. Parameters from Eqs. (1, 2), namely $m,\ c,\ k$ and (what is most important) c_j and k_j, describe mechanical system regardless of the excitation type

2. Nonlinear response of the modeled system occurs at oscillations with large enough amplitudes. Hence, if the set of equations (14) is found, *separately*, for system oscillating at, say, *two* different amplitudes – it might be possible (due to the nonlinearity) that these **2m complex** equations will be linearly independent. Since the number of **complex** unknowns ($\mathbf{m} + \mathbf{p_{damp}} + \mathbf{p_{stiff}}$) is kept constant - system of equations becomes overdetermined and thus may have solution[1].

[1] In this case it is probably fine to say that "overdetermined system of equations <u>has to</u> have solution". System of

To describe the idea mathematically, let us consider the same mechanical system subjected to multiple forcing functions, which caused oscillations at multiple different displacement magnitudes (e.g. multiple swept sines of different forcing level), see Eq. (15).

$$\begin{cases} m\ddot{x} + c\dot{x} + kx + c_2\dot{x}|\dot{x}| + c_3\dot{x}|\dot{x}|^2 + & \cdots + k_2x|x| + k_3x|x|^2 + \cdots = f_I(t) \\ m\ddot{x} + c\dot{x} + kx + c_2\dot{x}|\dot{x}| + c_3\dot{x}|\dot{x}|^2 + & \cdots + k_2x|x| + k_3x|x|^2 + \cdots = f_{II}(t) \\ & \vdots \\ m\ddot{x} + c\dot{x} + kx + c_2\dot{x}|\dot{x}| + c_3\dot{x}|\dot{x}|^2 + & \cdots + k_2x|x| + k_3x|x|^2 + \cdots = f_r(t) \end{cases}, \tag{15}$$

where r is the number of different forcing functions used in exciting the mechanical system.

If we repeat the derivation presented above in this section we end up with r-times the number of equations and the same number of unknowns (see Eq. 16).

$$\begin{bmatrix} S^{\,1}_{XF,I} & & & S^{\,1}_{dY_2F,I} & S^{\,1}_{dY_3F,I} & & S^{\,1}_{Y_2F,I} & S^{\,1}_{Y_3F,I} & \\ & \ddots & & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \\ & & S^{\,m}_{XF,I} & S^{\,m}_{dY_2F,I} & S^{\,m}_{dY_3F,I} & & S^{\,1}_{Y_2F,I} & S^{\,1}_{Y_3F,I} & \\ S^{\,1}_{XF,II} & & & S^{\,1}_{dY_2F,II} & S^{\,1}_{dY_3F,II} & & S^{\,1}_{Y_2F,II} & S^{\,1}_{Y_3F,II} & \\ & \ddots & & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \\ & & S^{\,m}_{XF,II} & S^{\,m}_{dY_2F,II} & S^{\,m}_{dY_3F,II} & & S^{\,1}_{Y_2F,II} & S^{\,1}_{Y_3F,II} & \\ & \vdots & & & & \ddots & & & \ddots \\ S^{\,1}_{XF,r} & & & S^{\,1}_{dY_2F,r} & S^{\,1}_{dY_3F,r} & & S^{\,1}_{Y_2F,r} & S^{\,1}_{Y_3F,r} & \\ & \ddots & & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \\ & & S^{\,m}_{XF,r} & S^{\,m}_{dY_2F,r} & S^{\,m}_{dY_3F,r} & & S^{\,1}_{Y_2F,r} & S^{\,1}_{Y_3F,r} & \end{bmatrix} \begin{bmatrix} D_1 \\ \vdots \\ D_m \\ c_2 \\ c_3 \\ \vdots \\ k_2 \\ k_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} S^{\,1}_{FF,I} \\ \vdots \\ S^{\,m}_{FF,I} \\ S^{\,1}_{FF,II} \\ \vdots \\ S^{\,m}_{FF,II} \\ \vdots \\ S^{\,1}_{FF,r} \\ \vdots \\ S^{\,m}_{FF,r} \end{bmatrix} \tag{16}$$

Since some of the parameters in Eq. (16) are complex, the estimates of c_j's and k_j's are not guaranteed to be real numbers. To overcome this issue, real and imaginary parts of the unknowns should be estimated separately – what would result in enforcing nonlinear parameters values to be real. To do so, Eq. (16) should be brought to its equivalent form presented in Eq. (17). The system of derived equations (17) is now overdetermined and the unknown parameters can be estimated via solving e.g. least squares optimization problem.

equations (does not matter if it is overdetermined or determined) was obtained in the numerical/experimental tests of existing mechanical system - thus solution has to exist as well. To be more precise - if the model for nonlinearity was guessed correctly, then the system of equations (possibly overdetermined) has solution.

$$
\begin{bmatrix}
Re\{S^1_{XF,I}\} & -Im\{S^1_{XF,I}\} & & & & Re\{S^1_{dY_2F,I}\} & & Re\{S^1_{Y_2F,I}\} & \\
Im\{S^1_{XF,I}\} & Re\{S^1_{XF,I}\} & & & & Im\{S^1_{dY_2F,I}\} & & Im\{S^1_{Y_2F,I}\} & \\
& & \ddots & & & \vdots & \cdots & \vdots & \cdots \\
& & & Re\{S^m_{XF,I}\} & -Im\{S^m_{XF,I}\} & Re\{S^m_{dY_2F,I}\} & & Re\{S^1_{Y_2F,I}\} & \\
& & & Im\{S^m_{XF,I}\} & Re\{S^m_{XF,I}\} & Im\{S^m_{dY_2F,I}\} & & Im\{S^1_{Y_2F,I}\} & \\
Re\{S^1_{XF,II}\} & -Im\{S^1_{XF,II}\} & & & & Re\{S^1_{dY_2F,II}\} & & Re\{S^1_{Y_2F,II}\} & \\
Im\{S^1_{XF,II}\} & Re\{S^1_{XF,II}\} & & & & Im\{S^1_{dY_2F,II}\} & & Im\{S^1_{Y_2F,II}\} & \\
& & \ddots & & & \vdots & \cdots & \vdots & \cdots \\
& & & Re\{S^m_{XF,II}\} & -Im\{S^m_{XF,II}\} & Re\{S^m_{dY_2F,II}\} & & Re\{S^1_{Y_2F,II}\} & \\
& & & Im\{S^m_{XF,II}\} & Re\{S^m_{XF,II}\} & Im\{S^m_{dY_2F,II}\} & & Im\{S^1_{Y_2F,II}\} & \\
& & \vdots & & & & \ddots & & \ddots \\
Re\{S^1_{XF,r}\} & -Im\{S^1_{XF,r}\} & & & & Re\{S^1_{dY_2F,r}\} & & Re\{S^1_{Y_2F,r}\} & \\
Im\{S^1_{XF,r}\} & Re\{S^1_{XF,r}\} & & & & Im\{S^1_{dY_2F,r}\} & & Im\{S^1_{Y_2F,r}\} & \\
& & \ddots & & & \vdots & \cdots & \vdots & \cdots \\
& & & Re\{S^m_{XF,r}\} & -Im\{S^m_{XF,r}\} & Re\{S^m_{dY_2F,r}\} & & Re\{S^1_{Y_2F,r}\} & \\
& & & Im\{S^m_{XF,r}\} & Re\{S^m_{XF,r}\} & Im\{S^m_{dY_2F,r}\} & & Im\{S^1_{Y_2F,r}\} &
\end{bmatrix}
\begin{bmatrix}
Re\{D_1\} \\ Im\{D_1\} \\ \vdots \\ Re\{D_m\} \\ Im\{D_m\} \\ c_2 \\ \vdots \\ k_2 \\ \vdots
\end{bmatrix}
=
\begin{bmatrix}
Re\{S^1_{FF,I}\} \\ Im\{S^1_{FF,I}\} \\ \vdots \\ Re\{S^m_{FF,I}\} \\ Im\{S^m_{FF,I}\} \\ Re\{S^1_{FF,II}\} \\ Im\{S^1_{FF,II}\} \\ \vdots \\ Re\{S^m_{FF,II}\} \\ Im\{S^m_{FF,II}\} \\ \vdots \\ Re\{S^1_{FF,r}\} \\ Im\{S^1_{FF,r}\} \\ \vdots \\ Re\{S^m_{FF,r}\} \\ Im\{S^m_{FF,r}\}
\end{bmatrix}
\tag{17}
$$

1.2.2 $\mathbf{H}_2$-NIXO

To identify the nonlinear system using the new $\mathbf{H}_2$ Nonlinear Identification through eXtended Outputs Estimator ($\mathbf{H}_2$-NIXO), bring Eq. (6) to a form of Eq. (18) and right-multiply it by matrix $\bigstar$ defined in Eq. (19).

Obtained in this way Eq. (20) is valid for every individual frequency sample. Thus, it is possible to express each of these equations in a matrix form shown in Eq. (21). Note that the frequency sample number was indicated in the parameters sub- or superscripts, e.g. $S_{XX}(\Omega_i) = S^{\,i}_{XX}$ or $D(\Omega_i) = D_i$.

$$
\begin{bmatrix} D(\Omega) & c_2 & c_3 & \ldots & k_2 & k_3 & \ldots \end{bmatrix}
\begin{bmatrix} \mathbf{X} \\ d\mathbf{Y}_2 \\ d\mathbf{Y}_3 \\ \vdots \\ \mathbf{Y}_2 \\ \mathbf{Y}_3 \\ \vdots \end{bmatrix}
= \mathbf{F} \qquad \Bigg| \bigstar
\tag{18}
$$

$$
\bigstar = \begin{bmatrix} \mathbf{X}^H & d\mathbf{Y}_2^H & d\mathbf{Y}_3^H & \ldots & \mathbf{Y}_2^H & \mathbf{Y}_3^H & \ldots \end{bmatrix}
\tag{19}
$$

$$
\begin{bmatrix} D(\Omega) & c_2 & c_3 & \ldots & k_2 & k_3 & \ldots \end{bmatrix}
\begin{bmatrix}
S_{XX} & S_{XdY_2} & S_{XdY_3} & \ldots & S_{XY_2} & S_{XY_3} & \ldots \\
S_{dY_2X} & S_{dY_2dY_2} & S_{dY_2dY_3} & \ldots & S_{dY_2Y_2} & S_{dY_2Y_3} & \ldots \\
S_{dY_3X} & S_{dY_3dY_2} & S_{dY_3dY_3} & \ldots & S_{dY_3Y_2} & S_{dY_3Y_3} & \ldots \\
& & & \ddots & & & \ddots \\
S_{Y_2X} & S_{Y_2dY_2} & S_{Y_2dY_3} & \ldots & S_{Y_2Y_2} & S_{Y_2Y_3} & \ldots \\
S_{Y_3X} & S_{Y_3dY_2} & S_{Y_3dY_3} & \ldots & S_{Y_3Y_2} & S_{Y_3Y_3} & \ldots \\
& & & \ddots & & & \ddots
\end{bmatrix}
=
$$

$$
= \begin{bmatrix} S_{FX} & S_{FdY_2} & S_{FdY_3} & \ldots & S_{FY_2} & S_{FY_3} & \ldots \end{bmatrix} \Longrightarrow
$$

$$
\Longrightarrow
\begin{bmatrix}
S_{XX} & S_{dY_2X} & S_{dY_3X} & \dots & S_{Y_2X} & S_{Y_3X} & \dots \\
S_{XdY_2} & S_{dY_2dY_2} & S_{dY_3dY_2} & \dots & S_{Y_2dY_2} & S_{Y_3dY_2} & \dots \\
S_{XdY_3} & S_{dY_2dY_3} & S_{dY_3dY_3} & \dots & S_{Y_2dY_3} & S_{Y_3dY_3} & \dots \\
 & & & \ddots & & & \ddots \\
S_{XY_2} & S_{dY_2Y_2} & S_{dY_3Y_2} & \dots & S_{Y_2Y_2} & S_{Y_3Y_2} & \dots \\
S_{XY_3} & S_{dY_2dY_3} & S_{dY_3Y_3} & \dots & S_{Y_2Y_3} & S_{Y_3Y_3} & \dots \\
 & & & \ddots & & & \ddots
\end{bmatrix}
\begin{bmatrix}
D(\Omega) \\ c_2 \\ c_3 \\ \vdots \\ k_2 \\ k_3 \\ \vdots
\end{bmatrix}
=
\begin{bmatrix}
S_{FX} \\ S_{FdY_2} \\ S_{FdY_3} \\ \vdots \\ S_{FY_2} \\ S_{FY_3} \\ \vdots
\end{bmatrix}
\tag{20}
$$

$$
\begin{bmatrix}
S^{\,1}_{XX} & & & S^{\,1}_{dY_2X} & S^{\,1}_{dY_3X} & & S^{\,1}_{Y_2X} & S^{\,1}_{Y_3X} & \\
 & \ddots & & & & \dots & & & \dots \\
 & & S^{\,m}_{XX} & S^{\,m}_{dY_2X} & S^{\,m}_{dY_3X} & & S^{\,m}_{Y_2X} & S^{\,m}_{Y_3X} & \\
S^{\,1}_{XdY_2} & & & S^{\,1}_{dY_2dY_2} & S^{\,1}_{dY_3dY_2} & & S^{\,1}_{Y_2dY_2} & S^{\,1}_{Y_3dY_2} & \\
 & \ddots & & & & \dots & & & \dots \\
 & & S^{\,m}_{XdY_2} & S^{\,m}_{dY_2dY_2} & S^{\,m}_{dY_3dY_2} & & S^{\,m}_{Y_2dY_2} & S^{\,m}_{Y_3dY_2} & \\
S^{\,1}_{XdY_3} & & & S^{\,1}_{dY_2dY_3} & S^{\,1}_{dY_3dY_3} & & S^{\,1}_{Y_2dY_3} & S^{\,1}_{Y_3dY_3} & \\
 & \ddots & & & & \dots & & & \dots \\
 & & S^{\,m}_{XdY_3} & S^{\,m}_{dY_2dY_3} & S^{\,m}_{dY_3dY_3} & & S^{\,m}_{Y_2dY_3} & S^{\,m}_{Y_3dY_3} & \\
 & \vdots & & & & \ddots & & & \ddots \\
S^{\,1}_{XY_2} & & & S^{\,1}_{dY_2Y_2} & S^{\,1}_{dY_3Y_2} & & S^{\,1}_{Y_2Y_2} & S^{\,1}_{Y_3Y_2} & \\
 & \ddots & & & & \dots & & & \dots \\
 & & S^{\,m}_{XY_2} & S^{\,m}_{dY_2Y_2} & S^{\,m}_{dY_3Y_2} & & S^{\,m}_{Y_2Y_2} & S^{\,m}_{Y_3Y_2} & \\
S^{\,1}_{XY_3} & & & S^{\,1}_{dY_2Y_3} & S^{\,1}_{dY_3Y_3} & & S^{\,1}_{Y_2Y_3} & S^{\,1}_{Y_3Y_3} & \\
 & \ddots & & & & \dots & & & \dots \\
 & & S^{\,m}_{XY_3} & S^{\,m}_{dY_2Y_3} & S^{\,m}_{dY_3Y_3} & & S^{\,m}_{Y_2Y_3} & S^{\,m}_{Y_3Y_3} & \\
 & & & & & \ddots & & & \ddots
\end{bmatrix}
\begin{bmatrix}
D_1 \\ \vdots \\ D_m \\ c_2 \\ c_3 \\ \vdots \\ k_2 \\ k_3 \\ \vdots
\end{bmatrix}
=
\begin{bmatrix}
S^{\,1}_{FX} \\ \vdots \\ S^{\,m}_{FX} \\ S^{\,1}_{FdY_2} \\ \vdots \\ S^{\,m}_{FdY_2} \\ S^{\,1}_{FdY_3} \\ \vdots \\ S^{\,m}_{FdY_3} \\ \vdots \\ S^{\,1}_{FY_2} \\ \vdots \\ S^{\,m}_{FY_2} \\ S^{\,1}_{FY_3} \\ \vdots \\ S^{\,m}_{FY_3} \\ \vdots
\end{bmatrix}
\tag{21}
$$

To assure that nonlinear parameters c_j and k_j are estimated as real numbers, analogically to the reasoning from section 1.2.1, let us bring the problem stated in Eq. (21) to its equivalent form for separated real and imaginary parts of the unknown parameters D_j, c_j and k_j, see Eq. (22).

$$
\left[\begin{array}{ccccccccc}
Re\{S^{1}_{XX}\} & -Im\{S^{1}_{XX}\} & & & & Re\{S^{1}_{dY_2X}\} & & Re\{S^{1}_{Y_2X}\} & \\
Im\{S^{1}_{XX}\} & Re\{S^{1}_{XX}\} & & & & Im\{S^{1}_{dY_2X}\} & & Im\{S^{1}_{Y_2X}\} & \\
 & & \ddots & & & & \cdots & & \cdots \\
 & & & Re\{S^{m}_{XX}\} & -Im\{S^{m}_{XX}\} & Re\{S^{m}_{dY_2X}\} & & Re\{S^{m}_{Y_2X}\} & \\
 & & & Im\{S^{m}_{XX}\} & Re\{S^{m}_{XX}\} & Im\{S^{m}_{dY_2X}\} & & Im\{S^{m}_{Y_2X}\} & \\
Re\{S^{1}_{XdY_2}\} & -Im\{S^{1}_{XdY_2}\} & & & & Re\{S^{1}_{dY_2dY_2}\} & & Re\{S^{1}_{Y_2dY_2}\} & \\
Im\{S^{1}_{XdY_2}\} & Re\{S^{1}_{XdY_2}\} & & & & Im\{S^{1}_{dY_2dY_2}\} & & Im\{S^{1}_{Y_2dY_2}\} & \\
 & & \ddots & & & & \cdots & & \cdots \\
 & & & Re\{S^{m}_{XdY_2}\} & -Im\{S^{m}_{XdY_2}\} & Re\{S^{m}_{dY_2dY_2}\} & & Re\{S^{m}_{Y_2dY_2}\} & \\
 & & & Im\{S^{m}_{XdY_2}\} & Re\{S^{m}_{XdY_2}\} & Im\{S^{m}_{dY_2dY_2}\} & & Im\{S^{m}_{Y_2dY_2}\} & \\
Re\{S^{1}_{XdY_3}\} & -Im\{S^{1}_{XdY_3}\} & & & & Re\{S^{1}_{dY_2dY_3}\} & & Re\{S^{1}_{Y_2dY_3}\} & \\
Im\{S^{1}_{XdY_3}\} & Re\{S^{1}_{XdY_3}\} & & & & Im\{S^{1}_{dY_2dY_3}\} & & Im\{S^{1}_{Y_2dY_3}\} & \\
 & & \ddots & & & & \cdots & & \cdots \\
 & & & Re\{S^{m}_{XdY_3}\} & -Im\{S^{m}_{XdY_3}\} & Re\{S^{m}_{dY_2dY_3}\} & & Re\{S^{m}_{Y_2dY_3}\} & \\
 & & & Im\{S^{m}_{XdY_3}\} & Re\{S^{m}_{XdY_3}\} & Im\{S^{m}_{dY_2dY_3}\} & & Im\{S^{m}_{Y_2dY_3}\} & \\
 & & \vdots & & & & \ddots & & \ddots \\
Re\{S^{1}_{XY_2}\} & -Im\{S^{1}_{XY_2}\} & & & & Re\{S^{1}_{dY_2Y_2}\} & & Re\{S^{1}_{Y_2Y_2}\} & \\
Im\{S^{1}_{XY_2}\} & Re\{S^{1}_{XY_2}\} & & & & Im\{S^{1}_{dY_2Y_2}\} & & Im\{S^{1}_{Y_2Y_2}\} & \\
 & & \ddots & & & & \cdots & & \cdots \\
 & & & Re\{S^{m}_{XY_2}\} & -Im\{S^{m}_{XY_2}\} & Re\{S^{m}_{dY_2Y_2}\} & & Re\{S^{m}_{Y_2dY_2}\} & \\
 & & & Im\{S^{m}_{XY_2}\} & Re\{S^{m}_{XY_2}\} & Im\{S^{m}_{dY_2Y_2}\} & & Im\{S^{m}_{Y_2dY_2}\} & \\
Re\{S^{1}_{XY_3}\} & -Im\{S^{1}_{XY_3}\} & & & & Re\{S^{1}_{dY_2Y_3}\} & & Re\{S^{1}_{Y_2Y_3}\} & \\
Im\{S^{1}_{XY_3}\} & Re\{S^{1}_{XY_3}\} & & & & Im\{S^{1}_{dY_2Y_3}\} & & Im\{S^{1}_{Y_2Y_3}\} & \\
 & & \ddots & & & & \cdots & & \cdots \\
 & & & Re\{S^{m}_{XY_3}\} & -Im\{S^{m}_{XY_3}\} & Re\{S^{m}_{dY_2Y_3}\} & & Re\{S^{m}_{Y_2dY_3}\} & \\
 & & & Im\{S^{m}_{XY_3}\} & Re\{S^{m}_{XY_3}\} & Im\{S^{m}_{dY_2Y_3}\} & & Im\{S^{m}_{Y_2dY_3}\} & \\
 & & & & & & \ddots & & \ddots
\end{array}\right]
\left[\begin{array}{c}
Re\{D_1\} \\ Im\{D_1\} \\ \vdots \\ Re\{D_m\} \\ Im\{D_m\} \\ c_2 \\ \vdots \\ k_2 \\ \vdots
\end{array}\right]
=
\left[\begin{array}{c}
Re\{S^{1}_{FX}\} \\ Im\{S^{1}_{FX}\} \\ \vdots \\ Re\{S^{m}_{FX}\} \\ Im\{S^{m}_{FX}\} \\
Re\{S^{1}_{FdY_2}\} \\ Im\{S^{1}_{FdY_2}\} \\ \vdots \\ Re\{S^{m}_{FdY_2}\} \\ Im\{S^{m}_{FdY_2}\} \\
Re\{S^{1}_{FdY_3}\} \\ Im\{S^{1}_{FdY_3}\} \\ \vdots \\ Re\{S^{m}_{FdY_3}\} \\ Im\{S^{m}_{FdY_3}\} \\
\vdots \\
Re\{S^{1}_{FY_2}\} \\ Im\{S^{1}_{FY_2}\} \\ \vdots \\ Re\{S^{m}_{FY_2}\} \\ Im\{S^{m}_{FY_2}\} \\
Re\{S^{1}_{FY_3}\} \\ Im\{S^{1}_{FY_3}\} \\ \vdots \\ Re\{S^{m}_{FY_3}\} \\ Im\{S^{m}_{FY_3}\} \\
\vdots
\end{array}\right]
\tag{22}
$$

Equation (22) shows that straight-forward application of the $\mathbf{H}_2$-NIXO to a nonlinear system results in obtaining overdetermined system of equations. If the unique solution of such system exists it can be estimated via solving e.g. least squares optimization problem.

1.3 Nonlinear Identification through eXtended Outputs Algorithms with Linear Data Provided

Derivation of two new nonlinear estimators was presented in section 1.2. Besides finding values of the parameters describing nonlinearities, the algorithms return estimates of the linear Frequency Response Function (FRF) as well. Since the linear experimental vibration analysis can be considered today as well–established, we could actually treat the linear FRF values as known (they could be obtained e.g. in experimental tests, where the structure vibrates at low enough amplitudes). With such assumption, we could modify the final equations from section 1.2 by bringing the FRF terms to the RHS vector of known values. This simple observation reduces the number of unknowns – now the only unknown parameters are c_j's and k_j's. It also significantly reduces sizes of matrices in Eqs. (17) and (22), which will result in algorithms becoming more efficient from the computational view point.

1.3.1 H$_1$-NIXO with Linear Data Provided

If the linear Frequency Response Function is known then Eq. (16) can be brought to form shown in Eq. (23) where quantities corresponding to D_j's are now placed in the RHS vector of known values. Note that collecting data from multiple vibration tests is no longer needed. System of equations (23) is most likely overdetermined, since the number of frequency samples ($m \sim 1000$) is usually larger than the number of unknown polynomial terms ($p_{dapmp},\ p_{stiff} \sim 10$). In case of $p_{dapmp} + p_{stiff} > m$ (which is possible but unlikely), then the number of equations can be populated by providing data collected in vibration tests where the mechanical system oscillates at multiple different amplitudes (as explained in section 1.2.1).

$$\underbrace{\begin{bmatrix} S^{\ 1}_{dY_2F} & S^{\ 1}_{dY_3F} & & S^{\ 1}_{Y_2F} & S^{\ 1}_{Y_3F} & \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots \\ S^{\ m}_{dY_2F} & S^{\ m}_{dY_3F} & & S^{\ 1}_{Y_2F} & S^{\ 1}_{Y_3F} & \end{bmatrix}}_{\mathbf{A}_1} \begin{bmatrix} c_2 \\ c_3 \\ \vdots \\ k_2 \\ k_3 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} S^{\ 1}_{FF} - S^{\ 1}_{XF} D_1 \\ \vdots \\ S^{\ m}_{FF} - S^{\ m}_{XF} D_m \end{bmatrix}}_{\mathbf{b}_1} \tag{23}$$

To enforce the algorithm to estimate the nonlinear parameters c_j and k_j as real numbers, Eq. (23) should be brought to its equivalent form shown in Eq. (24); matrix A_1 and vector b_1 are defined in Eq. (23).

$$\begin{bmatrix} Re\{\mathbf{A}_1\} \\ Im\{\mathbf{A}_1\} \end{bmatrix} \begin{bmatrix} c_2 \\ c_3 \\ \vdots \\ k_2 \\ k_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} Re\{\mathbf{b}_1\} \\ Im\{\mathbf{b}_1\} \end{bmatrix} \tag{24}$$

1.3.2 H$_2$-NIXO with Linear Data Provided

Derivation steps presented in this section are analogous to those in section 1.3.1. If the linear Frequency Response Function is known then Eq. (21) can be brought to form shown in Eq. (25)

where quantities corresponding to D_j's are now placed in the vector on the RHS. System of equations (25) is always overdetermined.

$$\underbrace{\begin{bmatrix} S^{1}_{dY_2X} & S^{1}_{dY_3X} & & S^{1}_{Y_2X} & S^{1}_{Y_3X} & \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \\ S^{m}_{dY_2X} & S^{m}_{dY_3X} & & S^{m}_{Y_2X} & S^{m}_{Y_3X} & \\ & & & & & \\ S^{1}_{dY_2dY_2} & S^{1}_{dY_3dY_2} & & S^{1}_{Y_2dY_2} & S^{1}_{Y_3dY_2} & \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \\ S^{m}_{dY_2dY_2} & S^{m}_{dY_3dY_2} & & S^{m}_{Y_2dY_2} & S^{m}_{Y_3dY_2} & \\ & & & & & \\ S^{1}_{dY_2dY_3} & S^{1}_{dY_3dY_3} & & S^{1}_{Y_2dY_3} & S^{1}_{Y_3dY_3} & \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \\ S^{m}_{dY_2dY_3} & S^{m}_{dY_3dY_3} & & S^{m}_{Y_2dY_3} & S^{m}_{Y_3dY_3} & \\ & & & & & \\ & & \ddots & & & \ddots \\ & & & & & \\ S^{1}_{dY_2Y_2} & S^{1}_{dY_3Y_2} & & S^{1}_{Y_2Y_2} & S^{1}_{Y_3Y_2} & \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \\ S^{m}_{dY_2Y_2} & S^{m}_{dY_3Y_2} & & S^{m}_{Y_2Y_2} & S^{m}_{Y_3Y_2} & \\ & & & & & \\ S^{1}_{dY_2Y_3} & S^{1}_{dY_3Y_3} & & S^{1}_{Y_2Y_3} & S^{1}_{Y_3Y_3} & \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \\ S^{m}_{dY_2Y_3} & S^{m}_{dY_3Y_3} & & S^{m}_{Y_2Y_3} & S^{m}_{Y_3Y_3} & \\ & & & & & \\ & & \ddots & & & \ddots \end{bmatrix}}_{\mathbf{A}_2} \begin{bmatrix} c_2 \\ c_3 \\ \vdots \\ k_2 \\ k_3 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} S^{1}_{FX} - S^{1}_{XX}D_1 \\ \vdots \\ S^{m}_{FX} - S^{m}_{XX}D_m \\ \\ S^{1}_{FdY_2} - S^{1}_{XdY_2}D_1 \\ \vdots \\ S^{m}_{FdY_2} - S^{m}_{XdY_2}D_m \\ \\ S^{1}_{FdY_3} - S^{1}_{XdY_3}D_1 \\ \vdots \\ S^{m}_{FdY_3} - S^{m}_{XdY_3}D_m \\ \\ \vdots \\ \\ S^{1}_{FY_2} - S^{1}_{XY_2}D_1 \\ \vdots \\ S^{m}_{FY_2} - S^{m}_{XY_2}D_m \\ \\ S^{1}_{FY_3} - S^{1}_{XY_3}D_1 \\ \vdots \\ S^{m}_{FY_3} - S^{m}_{XY_3}D_m \\ \\ \vdots \end{bmatrix}}_{\mathbf{b}_2} \tag{25}$$

To enforce the algorithm to estimate the nonlinear parameters c_j and k_j as real numbers, Eq. (25) should be brought to its equivalent form shown in Eq. (26); matrix A_2 and vector b_2 are defined in Eq. (25).

$$\begin{bmatrix} Re\{\mathbf{A}_2\} \\ Im\{\mathbf{A}_2\} \end{bmatrix} \begin{bmatrix} c_2 \\ c_3 \\ \vdots \\ k_2 \\ k_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} Re\{\mathbf{b}_2\} \\ Im\{\mathbf{b}_2\} \end{bmatrix} \tag{26}$$

2 Case Study – SDOF Systems

Different SDOF mechanical systems will be used to illustrate the accuracy of estimators. Input and output signals were generated by exciting the structure with various forcing functions:

- Swept cosine – described in Eq. (27)
- Broad–band burst random input – Eq. (28)

The results were then compared with one another showing weak/strong sides of each method.

Definition of Swept Cosine Forcing Signal

$$f(t) = F\cos(\Omega(t)\ t) \qquad \Omega(t) = \Omega_{st} + \frac{\Omega_{end} - \Omega_{st}}{t_{end} - t_{st}}(t - t_{st}) \qquad t \in [t_{st},\ t_{end}] \tag{27}$$

Definition of Broad-Band Burst Random Forcing Signal

$$f(t) = F\ BurstRand(t) \qquad t \in [t_{st},\ t_{end}] \tag{28}$$

2.1 Example 1

The algorithms are first evaluated using the Duffing equation (29). Values describing the mechanical system were proposed in [2] and are given in Tab. 1. Auto- and cross-spectra were obtained by applying 25-seconds-long Hanning windows with 51% of overlapping.

$$m\ddot{x} + c\dot{x} + kx + k_3x^3 = f(t) \tag{29}$$

Tab. 1: Parameters describing SDOF mechanical system with cubic stiffness nonlinearity. Example 1.

m $\left[\mathrm{kg}\right]$	c $\left[\frac{\mathrm{N\,s}}{\mathrm{m}}\right]$	k $\left[\frac{\mathrm{N}}{\mathrm{m}}\right]$	k_3 $\left[\frac{\mathrm{N}}{\mathrm{m}^3}\right]$
0.0024	0.03	0.5	0.3

Forcing signals and different testing scenarios are defined in:

- Tab. 2 and Tab. 3 (swept cosine),
- Tab. 14 and Tab. 15 (burst random).

2.1.1 Forcing Type: Swept Cosine

Tab. 2: Parameters describing swept cosine forcing function. Example 1.

F [N]	Ω_{st} [Hz]	Ω_{end} [Hz]	t_{st} [s]	t_{end} [s]
different values	0.01	6.28	0	1500

Tab. 3: Description of various scenarios the algorithms are tested with. Force amplitudes expressed in newtons, frequencies in hertz. Example 1.

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}	I/O Signals	Results
A101	81	0.01	0.001	-	-	-	-	0.01	6.28	Tab. 4	Tab. 5
A102		0.10	0.01							Tab. 6	Tab. 7
A103		0.25	0.01							Tab. 8	Tab. 9
A104		2.00	0.01							Tab. 10	Tab. 11
A105	83	0.25	0.01	6	-	-	-	0.01	6.28	Tab. 12	Tab. 13

DF – Decimation Factor

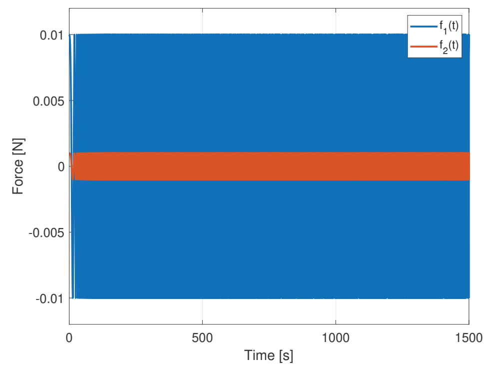
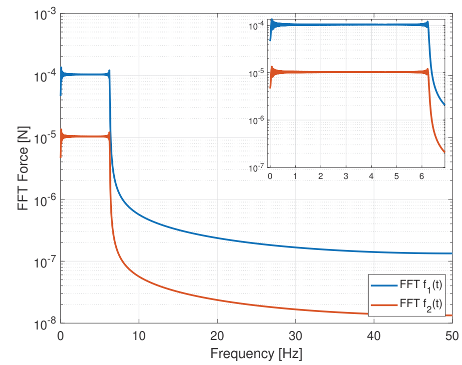
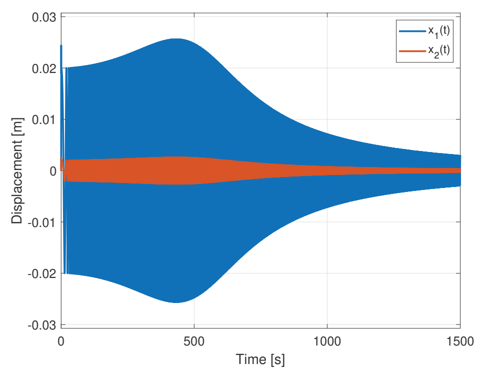
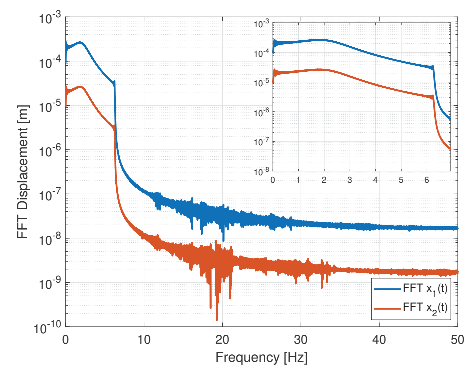
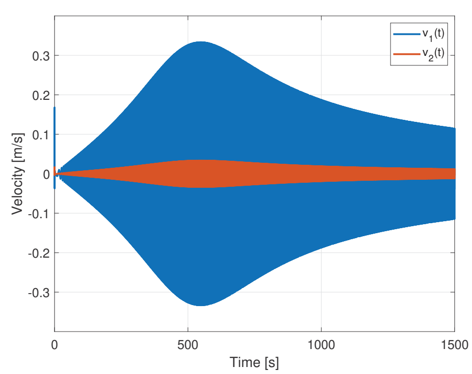
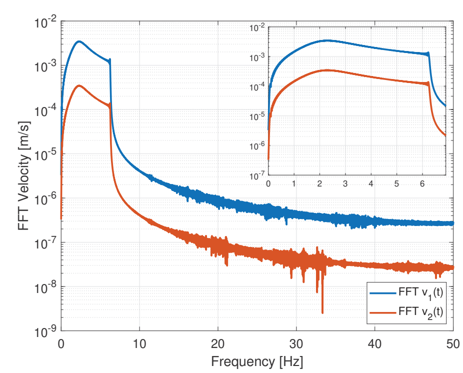
FO – Butterworth Filter Order

Ω^{filt} – cut-off frequency; Ω_1^{filt} and Ω_2^{filt} are lower and upper cut-off frequencies, respectively

Ω^{spect} – auto- and cross-spectra are computed for frequency range (Ω_1^{spect}, Ω_2^{spect}).

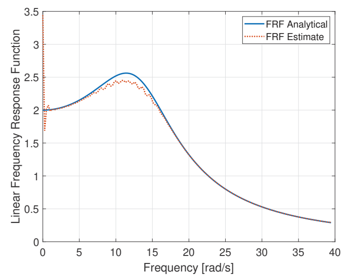
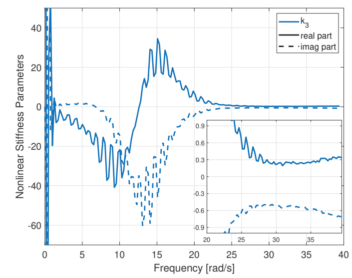
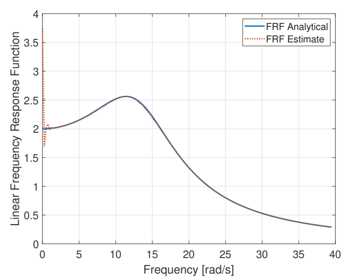
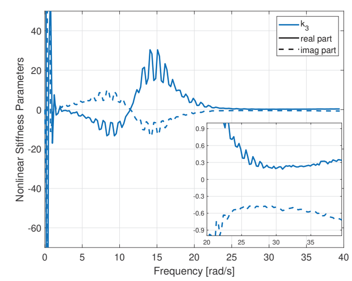
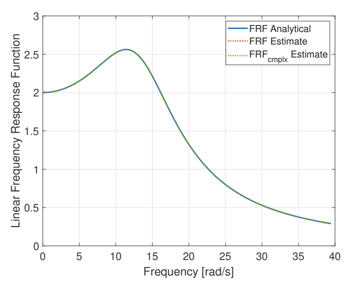
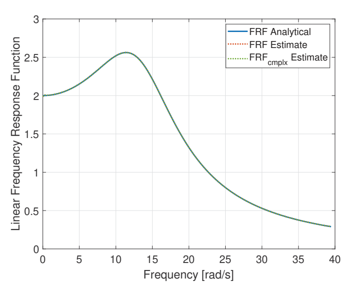
Tab. 4: SCENARIO A101: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A101	81	0.010	0.001	-	-	-	-	0.01	6.28

<table>
<tr><th>Signal</th><th>Time domain</th><th>Freq. domain</th></tr>
<tr><td>$f(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$x(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$v(t)$</td><td>
</td><td>
</td></tr>
</table>

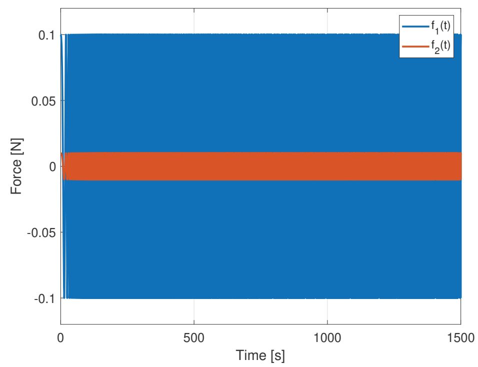
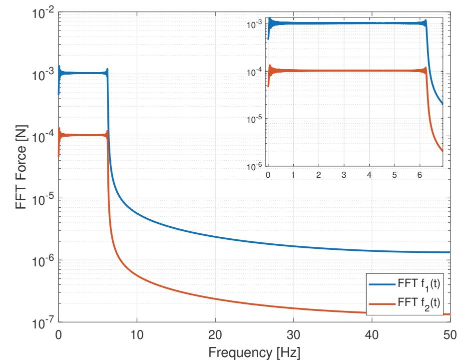
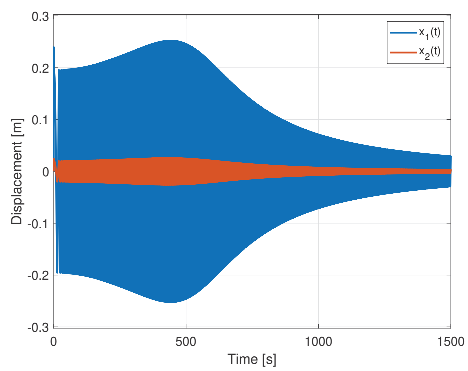
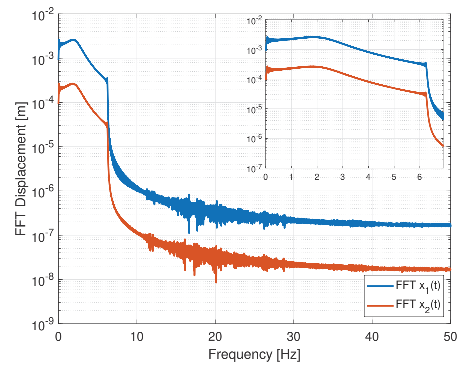
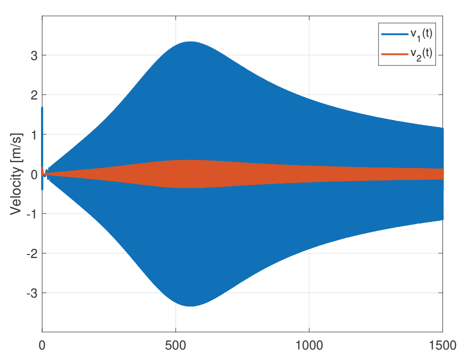
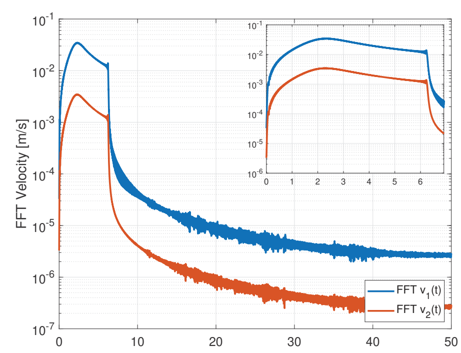
Tab. 5: SCENARIO A101: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A101	81	0.010	0.001	-	-	-	-	0.01	6.28

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
$\mathbf{H}_1$-NIFO		
$\mathbf{H}_2$-NIFO		
$\mathbf{H}_1$-NIXO		$k_3 = 0.29648$ (1.17%) $k_{3,cmplx} = 0.29648 - i0.00772$
$\mathbf{H}_2$-NIXO		$k_3 = 1.06363$ (254.54%) $k_{3,cmplx} = 1.06365 - i0.72434$
$\mathbf{H}_1$-NIXO with lin. data provided	$\times$	$k_3 = 0.46702$ (55.67%) $k_{3,cmplx} = 0.46702 + i0.52524$
$\mathbf{H}_2$-NIXO with lin. data provided	$\times$	$k_3 = 0.29538$ (1.54%) $k_{3,cmplx} = 0.29538 + i0.16784$

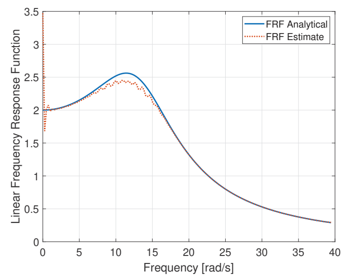
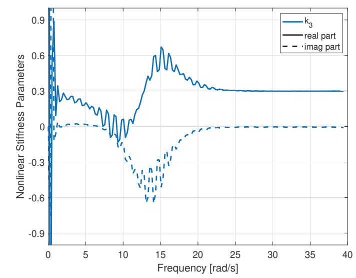
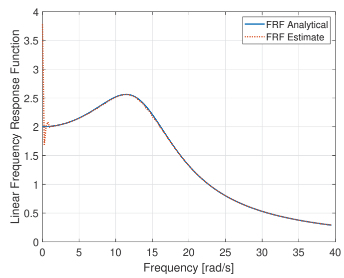
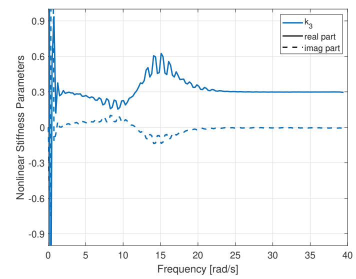
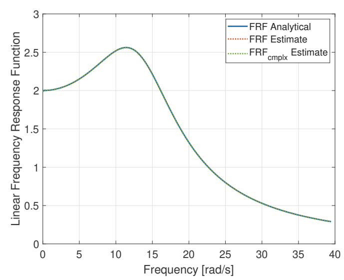
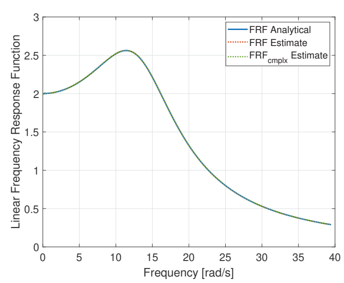
Tab. 6: SCENARIO A102: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A102	81	0.100	0.010	-	-	-	-	0.01	6.28

<table>
<tr><th>Signal</th><th>Time domain</th><th>Freq. domain</th></tr>
<tr><td>$f(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$x(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$v(t)$</td><td>
</td><td>
</td></tr>
</table>

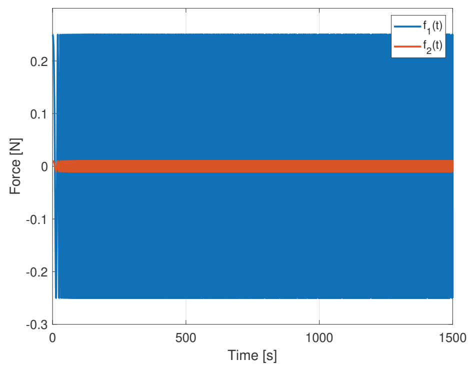
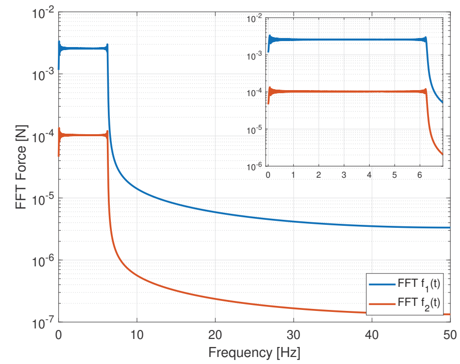
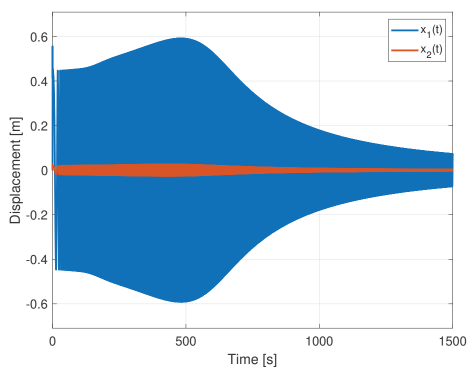
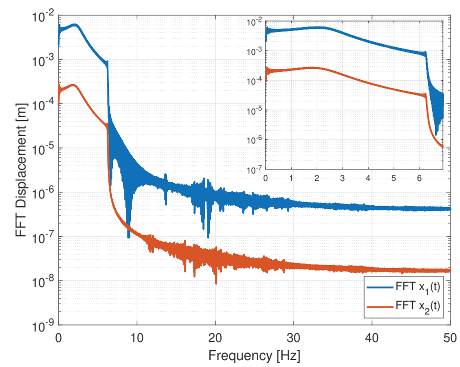
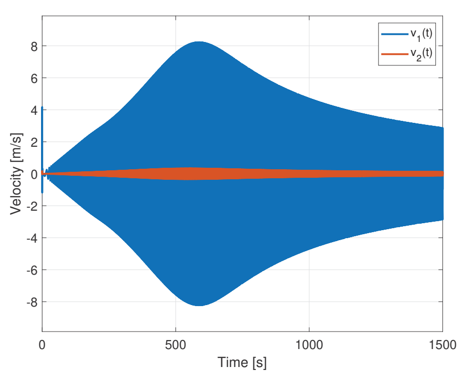
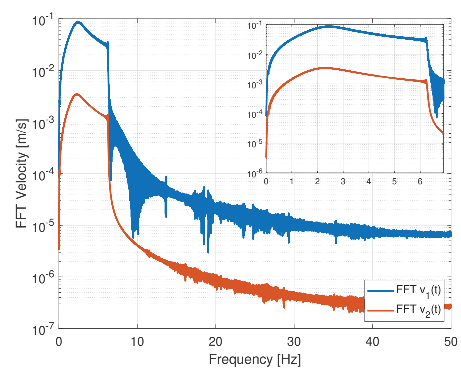
Tab. 7: SCENARIO A102: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A102	81	0.100	0.010	-	-	-	-	0.01	6.28

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
$\mathbf{H}_1$-NIFO		
$\mathbf{H}_2$-NIFO		
$\mathbf{H}_1$-NIXO		$k_3 = 0.29937$ (0.21%) $k_{3,cmplx} = 0.29937 - i0.00036$
$\mathbf{H}_2$-NIXO		$k_3 = 0.30706$ (2.35%) $k_{3,cmplx} = 0.30706 - i0.00652$
$\mathbf{H}_1$-NIXO with lin. data provided	$\times$	$k_3 = 0.30119$ (0.40%) $k_{3,cmplx} = 0.30119 + i0.00513$
$\mathbf{H}_2$-NIXO with lin. data provided	$\times$	$k_3 = 0.29943$ (0.19%) $k_{3,cmplx} = 0.29943 + i0.00139$

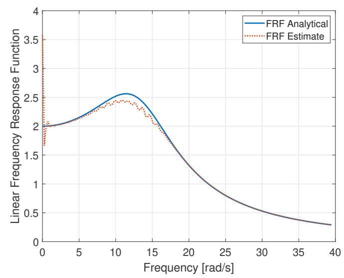
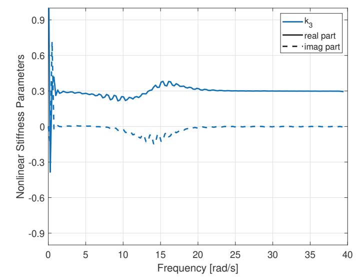
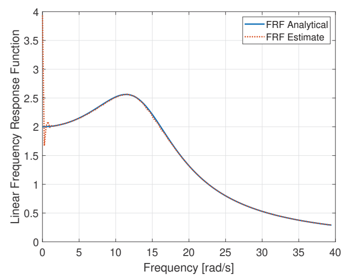
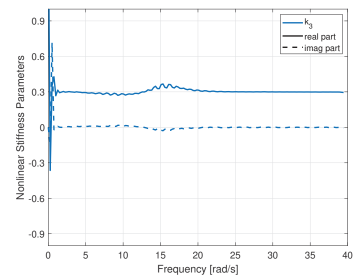
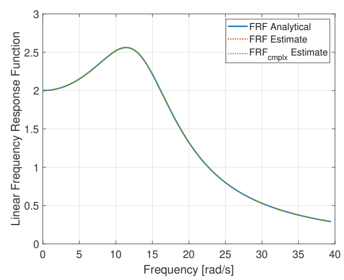
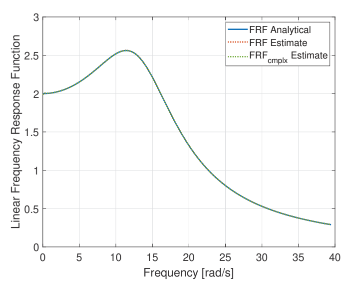
Tab. 8: SCENARIO A103: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A103	81	0.250	0.010	-	-	-	-	0.01	6.28

Signal	Time domain	Freq. domain
$f(t)$		
$x(t)$		
$v(t)$		

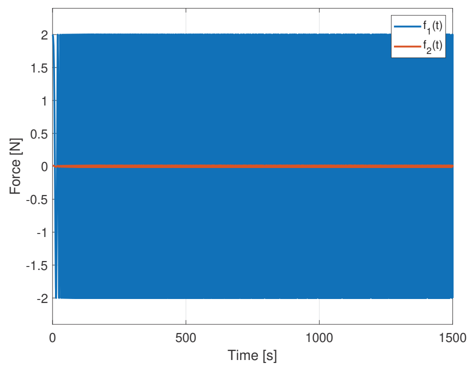
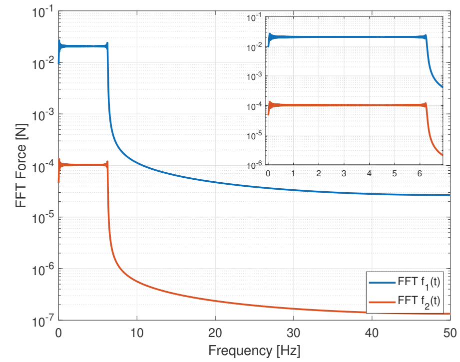
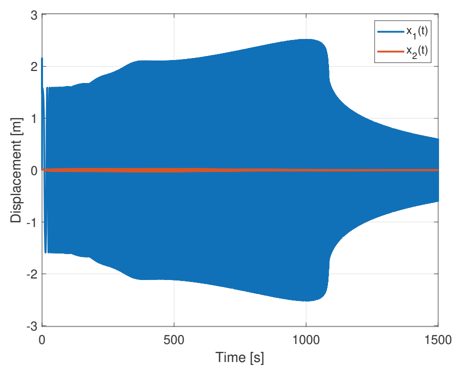
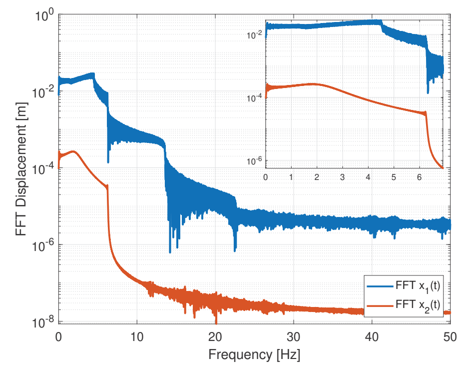
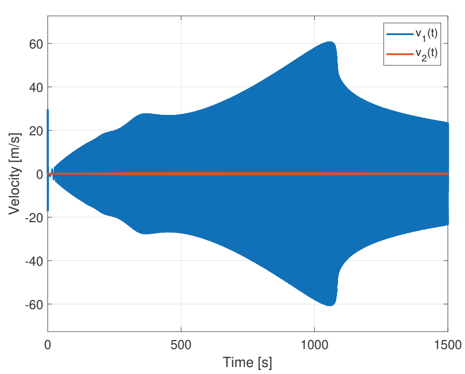
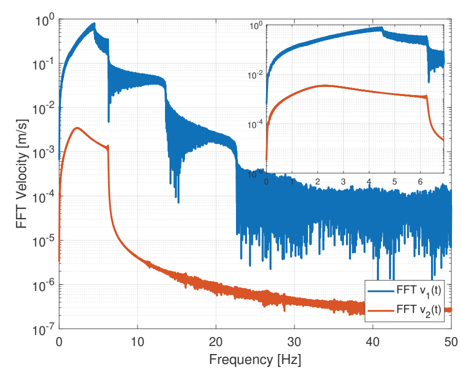
Tab. 9: SCENARIO A103: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A103	81	0.250	0.010	-	-	-	-	0.01	6.28

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 0.29939$ (0.20%) $k_{3,cmplx} = 0.29939 - i0.00033$
H$_2$-NIXO		$k_3 = 0.30068$ (0.23%) $k_{3,cmplx} = 0.30068 - i0.00103$
H$_1$-NIXO with lin. data provided	$\times$	$k_3 = 0.29975$ (0.08%) $k_{3,cmplx} = 0.29975 + i0.00071$
H$_2$-NIXO with lin. data provided	$\times$	$k_3 = 0.29944$ (0.19%) $k_{3,cmplx} = 0.29944 - i0.00001$

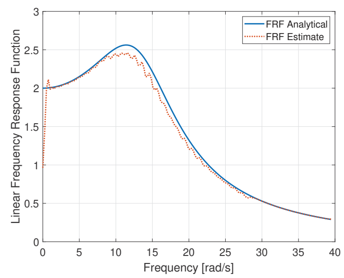
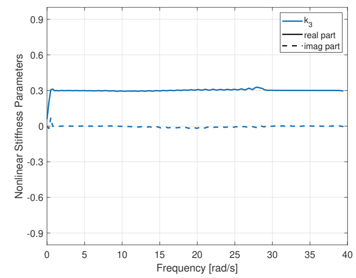
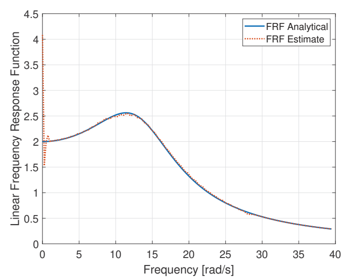
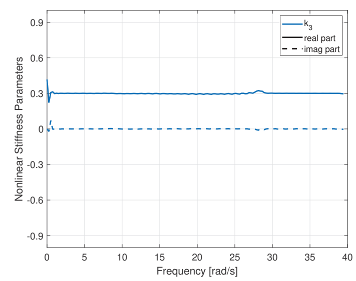
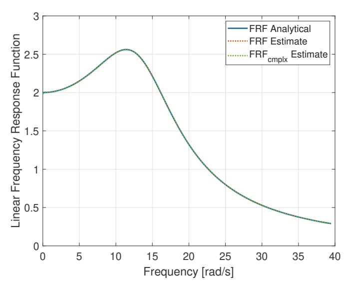
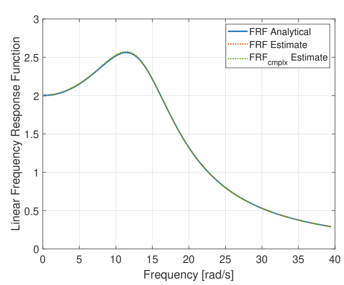
Tab. 10: SCENARIO A104: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A104	81	2.000	0.010	-	-	-	-	0.01	6.28

Signal	Time domain	Freq. domain
$f(t)$		
$x(t)$		
$v(t)$		

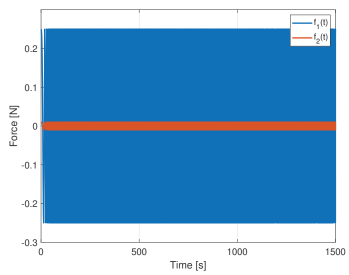
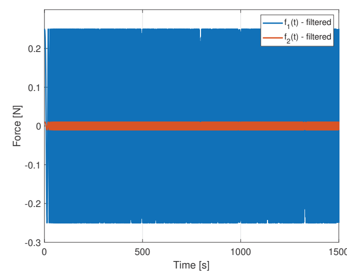
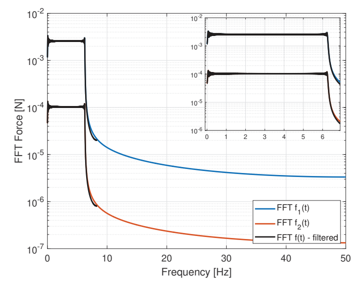
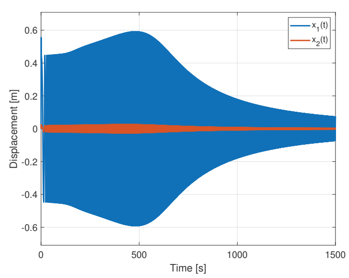
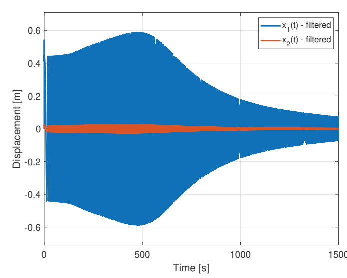
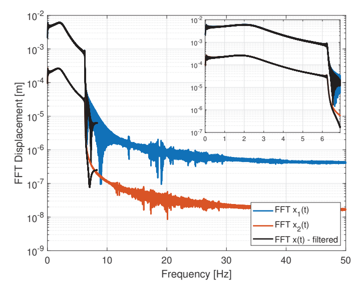
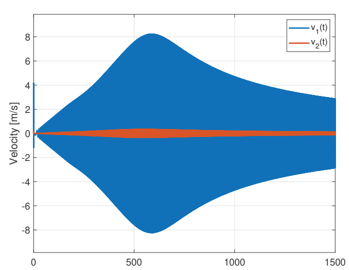
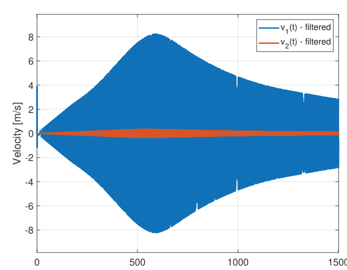
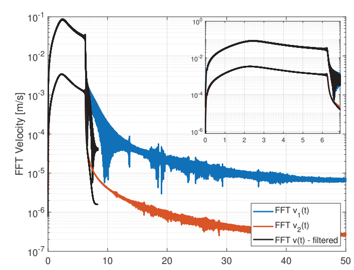
Tab. 11: SCENARIO A104: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A104	81	2.000	0.010	-	-	-	-	0.01	6.28

<table>
<tr><th>Method</th><th>Linear FRF estimate</th><th>Nonlin. param. estimates (rel. error)</th></tr>
<tr><td>H$_1$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_2$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_1$-NIXO</td><td>
</td><td>$k_3 = 0.29982\ (0.06\%)$
$k_{3,cmplx} = 0.29982 - i0.00009$</td></tr>
<tr><td>H$_2$-NIXO</td><td>
</td><td>$k_3 = 0.30056\ (0.19\%)$
$k_{3,cmplx} = 0.30056 + i0.00024$</td></tr>
<tr><td>H$_1$-NIXO with lin. data provided</td><td>×</td><td>$k_3 = 0.29981\ (0.06\%)$
$k_{3,cmplx} = 0.29981 + i0.00000$</td></tr>
<tr><td>H$_2$-NIXO with lin. data provided</td><td>×</td><td>$k_3 = 0.29978\ (0.07\%)$
$k_{3,cmplx} = 0.29978 - i0.00006$</td></tr>
</table>

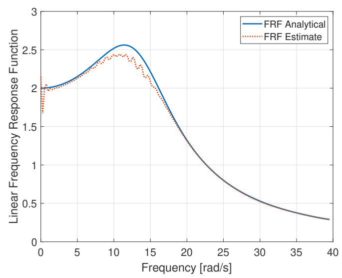
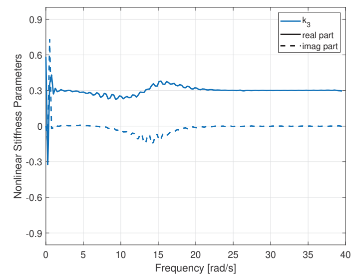
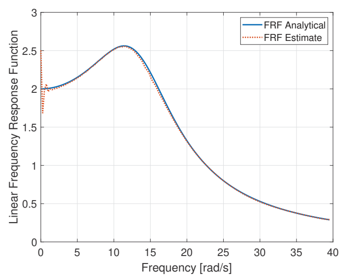
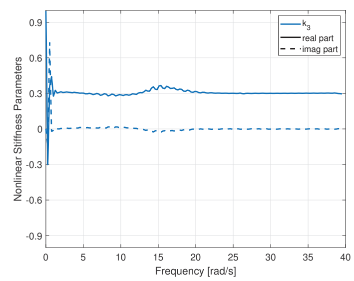
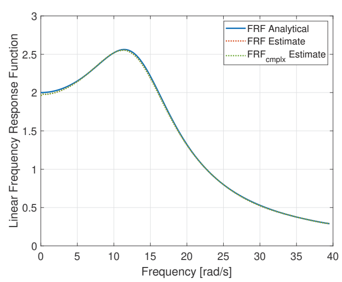
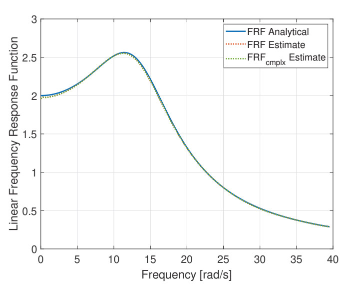
Tab. 12: SCENARIO A105: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A105	83	0.250	0.010	6	-	-	-	0.01	6.28

Signal	Time domain	Filtered signal	Freq. domain
$f(t)$			
$x(t)$			
$v(t)$			

Tab. 13: SCENARIO A105: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A105	83	0.250	0.010	6	-	-	-	0.01	6.28

<table>
<tr><th>Method</th><th>Linear FRF estimate</th><th>Nonlin. param. estimates (rel. error)</th></tr>
<tr><td>H$_1$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_2$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_1$-NIXO</td><td>
</td><td>$k_3 = 0.30427$ (1.42%)
$k_{3,cmplx} = 0.30427 - i0.00030$</td></tr>
<tr><td>H$_2$-NIXO</td><td>
</td><td>$k_3 = 0.30261$ (0.87%)
$k_{3,cmplx} = 0.30261 - i0.00092$</td></tr>
<tr><td>H$_1$-NIXO with lin. data provided</td><td>×</td><td>$k_3 = 0.30777$ (2.59%)
$k_{3,cmplx} = 0.30777 + i0.01029$</td></tr>
<tr><td>H$_2$-NIXO with lin. data provided</td><td>×</td><td>$k_3 = 0.30718$ (2.39%)
$k_{3,cmplx} = 0.30718 + i0.00909$</td></tr>
</table>

Comments

1. Each method estimates the linear frequency response function to a satisfactory extent. However, the $\mathbf{H}_1$-NIFO underestimates the FRF in the vicinity of its pick. This result was expected and is explained in [2].

2. If the I/O signals are collected in the tests where the mechanical system oscillates at low amplitudes (e.g. Scenario A101), most of the methods fail to estimate the nonlinear parameter. This result was expected since the nonlinearity does not make its presence felt when the system vibrates at low amplitudes.

3. When the system oscillates at larger amplitudes (Scenarios A102 and A103) the nonlinear parameter value is estimated with satisfactory precision. NIFO methods find k_3 values precisely for off-resonant frequencies, while for the vicinity of the resonance their estimations are erroneous.

4. Accuracy of the NIFO methods can be enhanced by providing I/O signals coming from tests where the structure vibrates at higher amplitudes (see Scenario A104).

5. The decimation factor does not affect estimators' accuracy if swept cosine is used as a forcing signal.

6. It is worth noting that (for the NIXO and NIXO –WLDP algorithms) the real part of k_3 found as a complex number matches the value of the cubic nonlinear parameter estimated as a real number. Moreover, the real part of k_3 is an accurate estimate when its imaginary part is found as zero (see Example 1), or at least is significantly lower than $Re\{k_3\}$ (Example 2). Results obtained with the NIFO algorithms show that $Im\{k_3\}$ is much lower than $Re\{k_3\}$ for off-resonant frequencies. Thus, it once again indicates that we should search for the accurate estimates of nonlinear parameters in the off-resonant frequency ranges.

2.1.2 Forcing Type: Broad-Band Burst Random

Tab. 14: Parameters describing broad band burst random forcing function and Butterworth filter used to process the input signal. Example 1.

F [N]	t_{st} [s]	t_{end} [s]	Δt [s]	Burst Start	Burst End
different values	0	512.5	0.01	0%	100%

Ω_1^{filt} [Hz]	Ω_2^{filt} [Hz]	Decimation Factor	Filter Order
different values		**different values**	

Tab. 15: Description of various scenarios the algorithms are tested with. Example 1.

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}	I/O Signals	Results
A201		0.1								Tab. 16	Tab. 17
A202		0.25								Tab. 18	Tab. 19
A203	121	0.5	0.01	-	5	0.3	6.28	0.3	6.28	Tab. 20	Tab. 21
A204		1								Tab. 22	Tab. 23
A205		2								Tab. 24	Tab. 25
A206		0.25								Tab. 26	Tab. 27
A207	124	0.5	0.01	3	7	0.3	6.28	0.3	6.28	Tab. 28	Tab. 29
A208		1								Tab. 30	Tab. 31
A209		0.25								Tab. 32	Tab. 33
A210	125	0.5	0.01	4	7	0.3	6.28	0.3	6.28	Tab. 34	Tab. 35
A211		1								Tab. 36	Tab. 37
A212		0.25								Tab. 38	Tab. 39
A213	126	0.5	0.01	5	7	0.3	6.28	0.3	6.28	Tab. 40	Tab. 41
A214		1								Tab. 42	Tab. 43

Force amplitudes expressed in newtons, frequencies in hertz.

DF – Decimation Factor

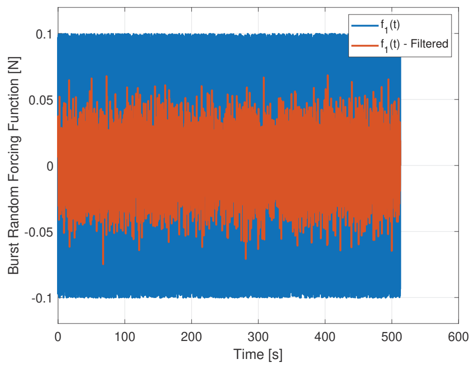
FO – Butterworth Filter Order

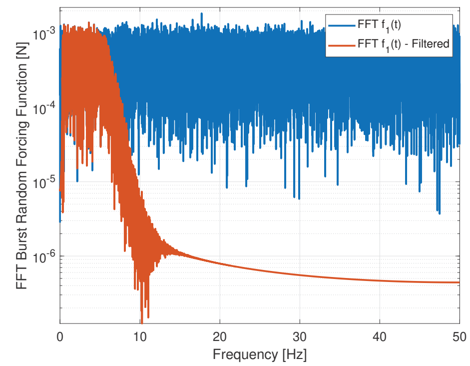
Ω^{filt} – cut-off frequency; Ω_1^{filt} and Ω_2^{filt} are lower and upper cut-off frequencies, respectively

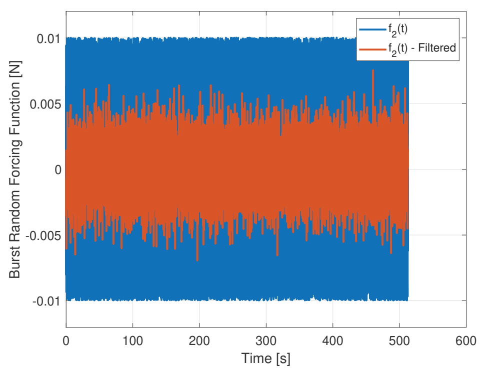
Ω^{spect} – auto- and cross-spectra are computed for frequency range (Ω_1^{spect}, Ω_2^{spect}).

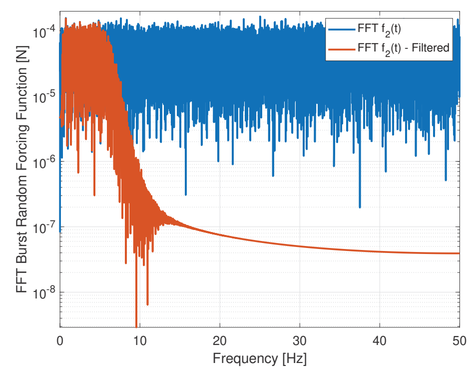
Tab. 16: SCENARIO A201: Input/Output Signals

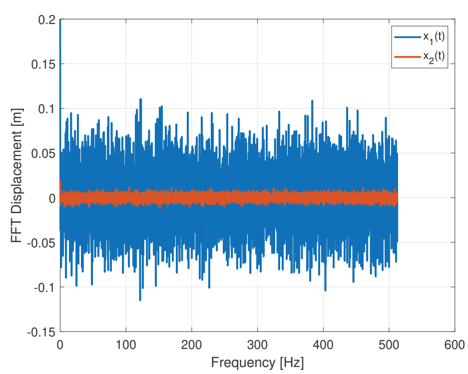
Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A201	121	0.100	0.010	-	5	0.3	6.28	0.3	6.28

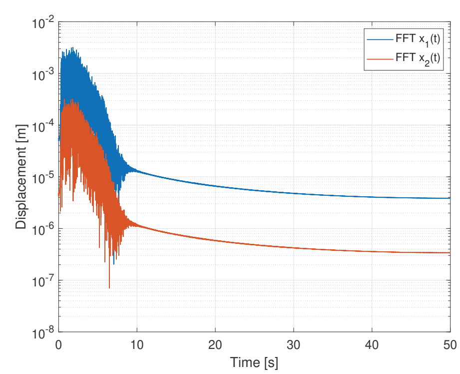
<table>
<tr><th>Signal</th><th>Time domain</th><th>Freq. domain</th></tr>
<tr><td>$f_I(t)$</td><td>


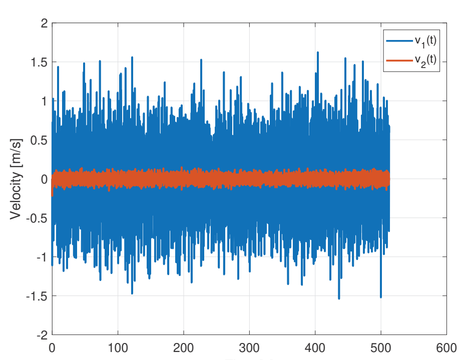
</td><td>


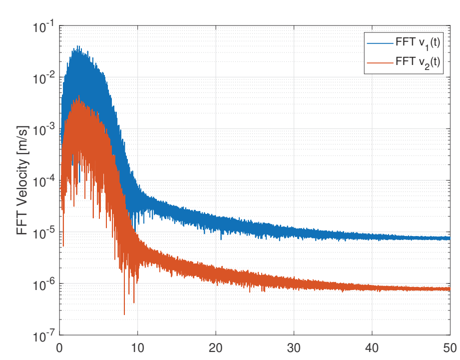
</td></tr>
<tr><td>$f_{II}(t)$</td><td>


</td><td>


</td></tr>
<tr><td>$x(t)$</td><td>


</td><td>


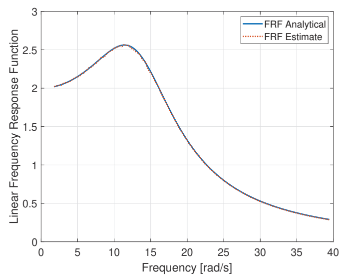
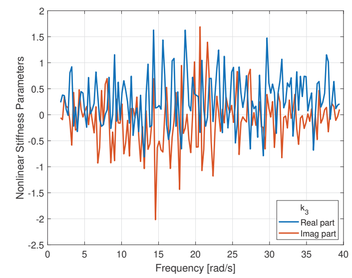
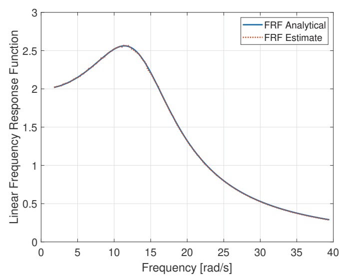
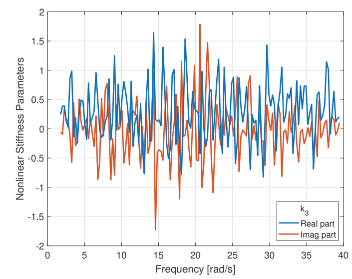
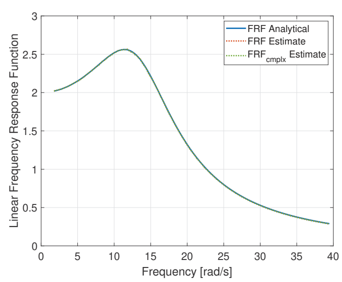
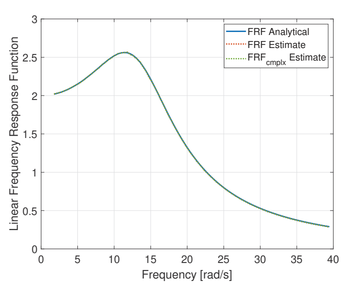
</td></tr>
<tr><td>$v(t)$</td><td>


</td><td>


</td></tr>
</table>

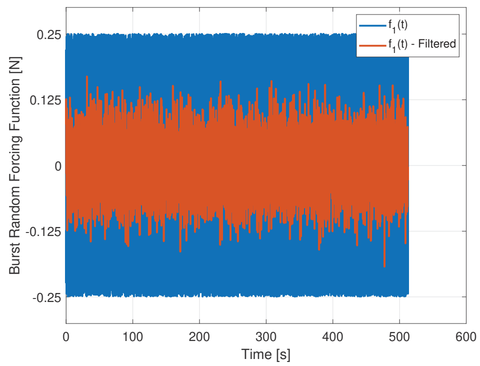
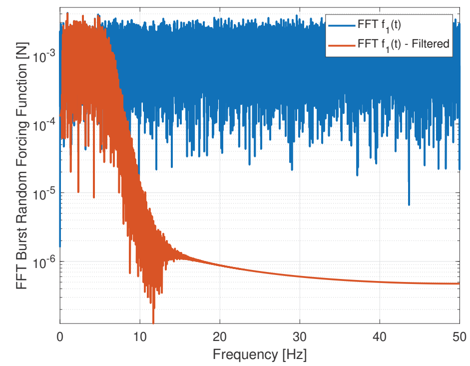
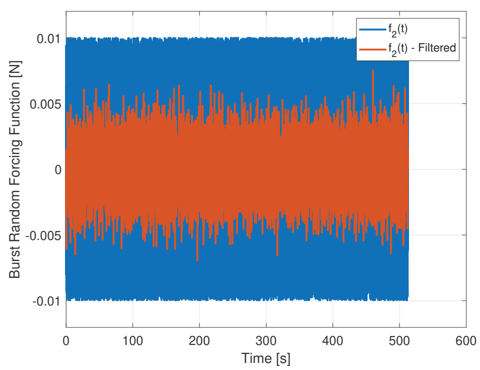
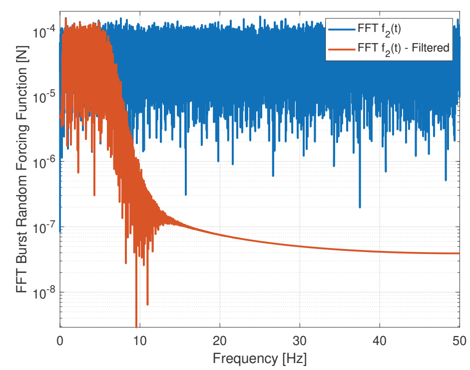
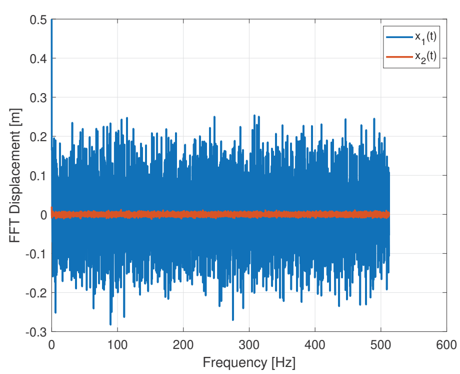
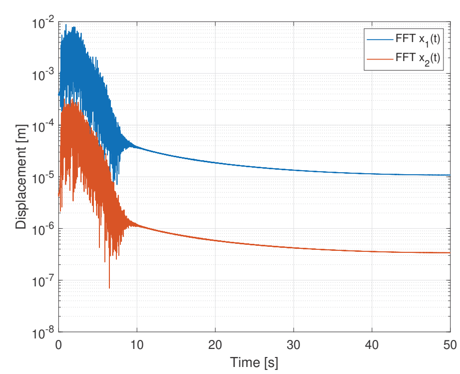
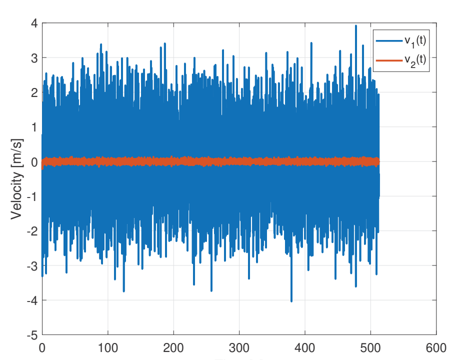
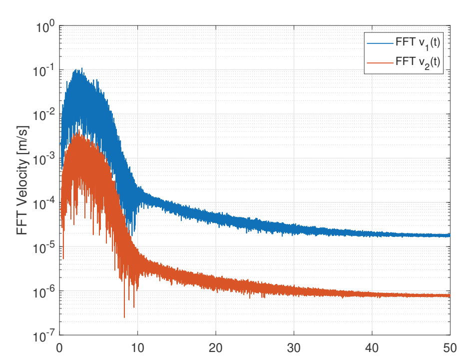
Tab. 17: SCENARIO A201: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A201	121	0.100	0.010	-	5	0.3	6.28	0.3	6.28

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 0.28429$ (5.24%) $k_{3,cmplx} = 0.28429 \ - \ i0.03650$
H$_2$-NIXO		$k_3 = 0.28671$ (4.43%) $k_{3,cmplx} = 0.28671 \ - \ i0.04621$
H$_1$-NIXO with lin. data provided	$\times$	$k_3 = 0.11068$ (63.11%) $k_{3,cmplx} = 0.11068 \ + \ i0.31514$
H$_2$-NIXO with lin. data provided	$\times$	$k_3 = 0.25714$ (14.29%) $k_{3,cmplx} = 0.25714 \ + \ i0.11933$

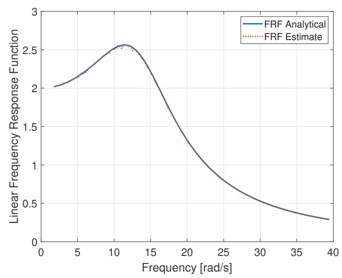
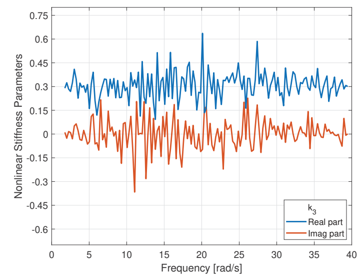
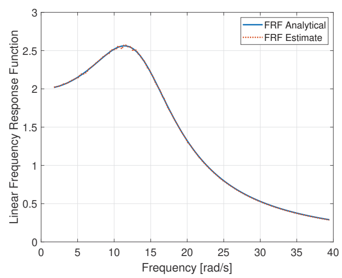
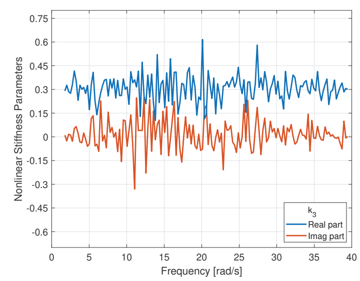
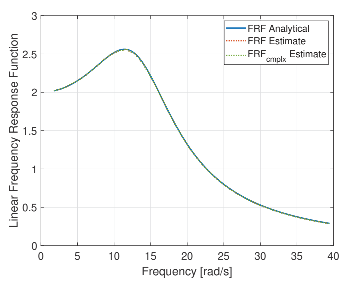
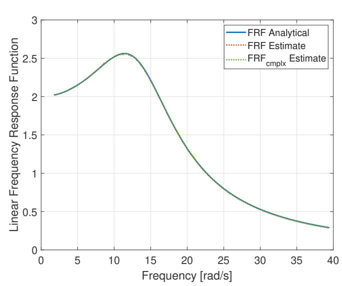
Tab. 18: SCENARIO A202: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A202	121	0.250	0.010	-	5	0.3	6.28	0.3	6.28

<table>
<tr><th>Signal</th><th>Time domain</th><th>Freq. domain</th></tr>
<tr><td>$f_I(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$f_{II}(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$x(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$v(t)$</td><td>
</td><td>
</td></tr>
</table>

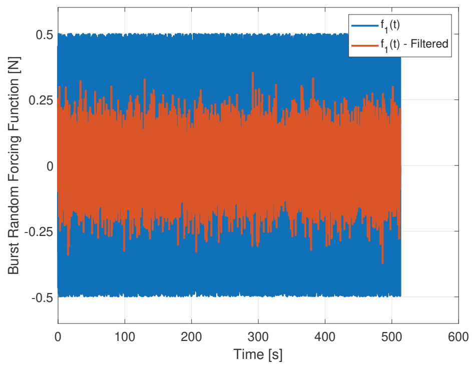
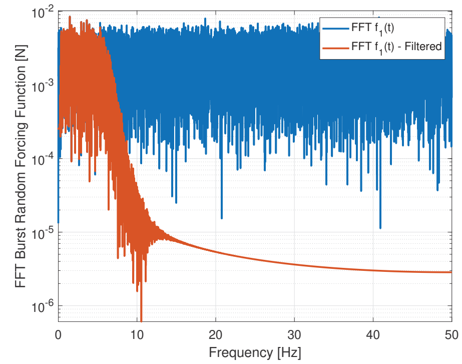
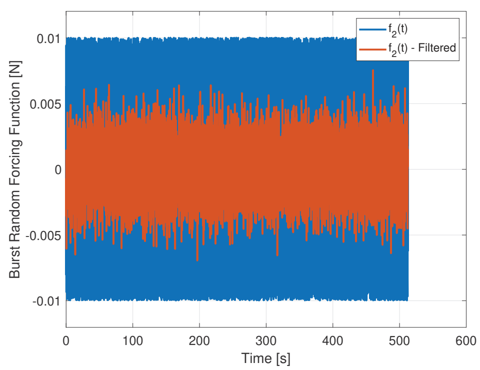
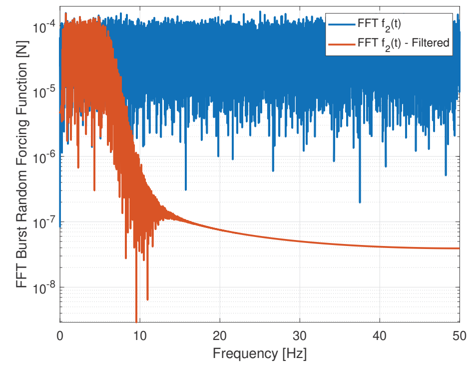
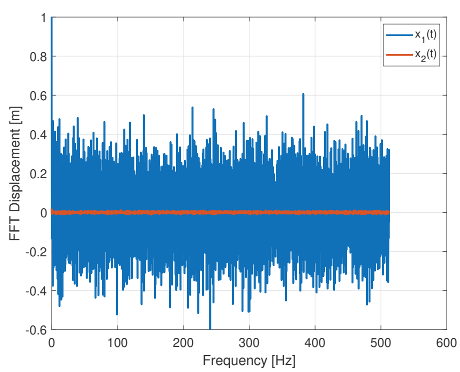
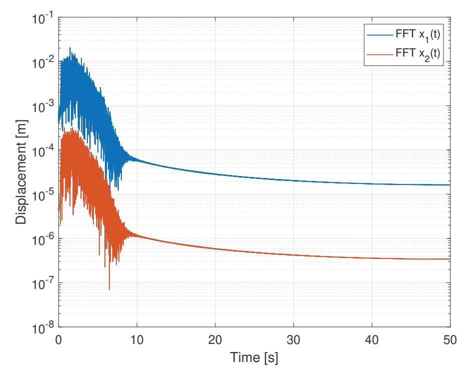
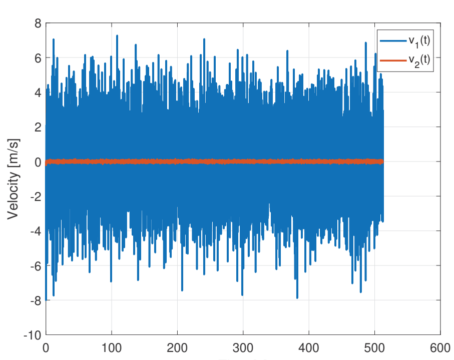
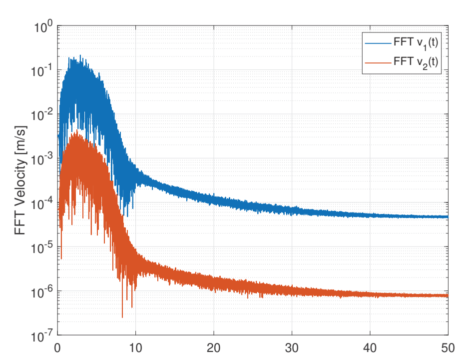
Tab. 19: SCENARIO A202: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A202	121	0.250	0.010	-	5	0.3	6.28	0.3	6.28

<table>
<tr><th>Method</th><th>Linear FRF estimate</th><th>Nonlin. param. estimates (rel. error)</th></tr>
<tr><td>H$_1$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_2$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_1$-NIXO</td><td>
</td><td>$k_3 = 0.30472$ (1.57%)
$k_{3,cmplx} = 0.30472 + i0.00217$</td></tr>
<tr><td>H$_2$-NIXO</td><td>
</td><td>$k_3 = 0.30541$ (1.80%)
$k_{3,cmplx} = 0.30541 + i0.01665$</td></tr>
<tr><td>H$_1$-NIXO with lin. data provided</td><td>×</td><td>$k_3 = 0.27793$ (7.36%)
$k_{3,cmplx} = 0.27793 + i0.05835$</td></tr>
<tr><td>H$_2$-NIXO with lin. data provided</td><td>×</td><td>$k_3 = 0.30220$ (0.73%)
$k_{3,cmplx} = 0.30220 + i0.03501$</td></tr>
</table>

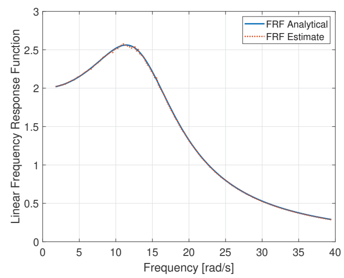
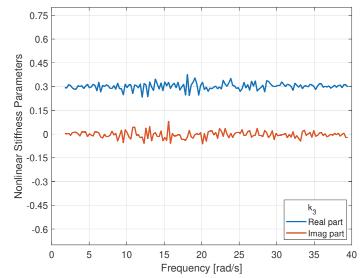
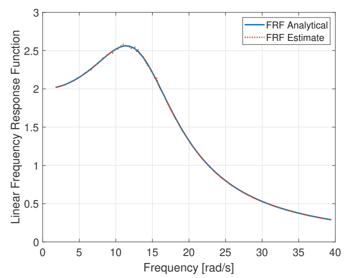
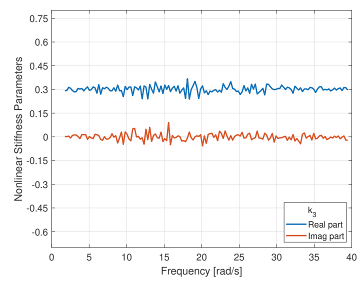
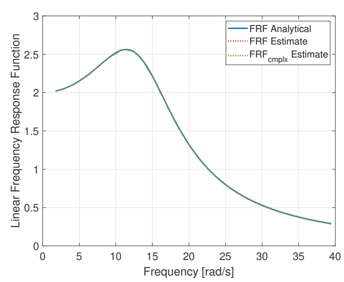
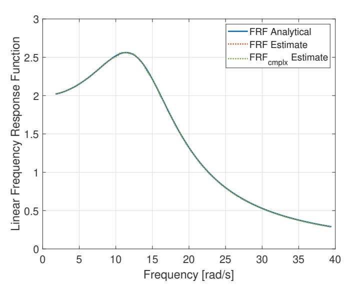
Tab. 20: SCENARIO A203: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A203	121	0.500	0.010	-	5	0.3	6.28	0.3	6.28

<table>
<tr><th>Signal</th><th>Time domain</th><th>Freq. domain</th></tr>
<tr><td>$f_I(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$f_{II}(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$x(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$v(t)$</td><td>
</td><td>
</td></tr>
</table>

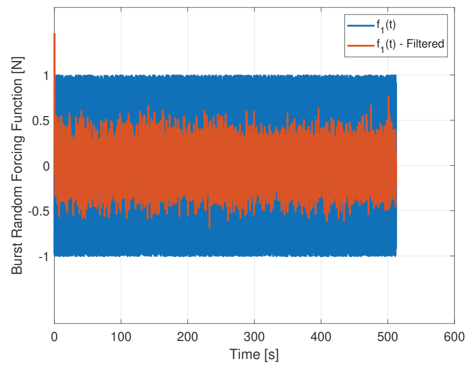
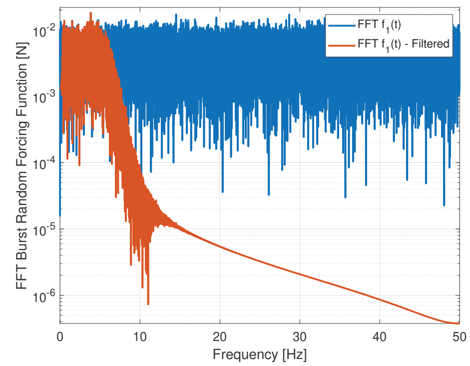
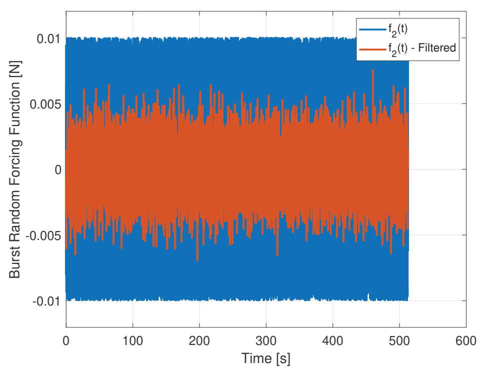
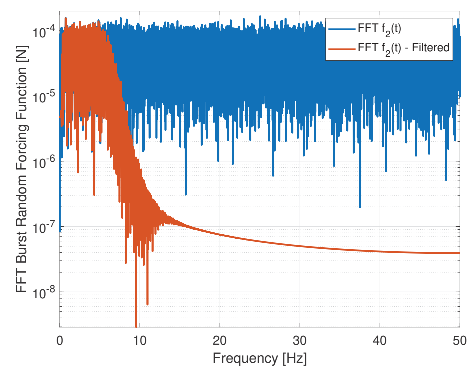
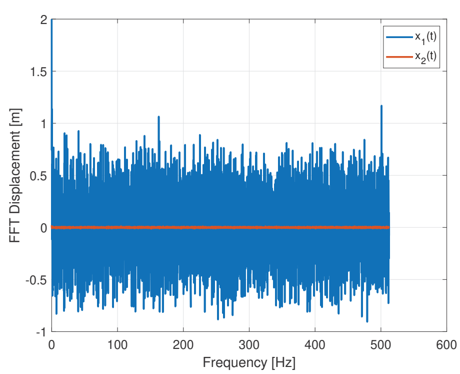
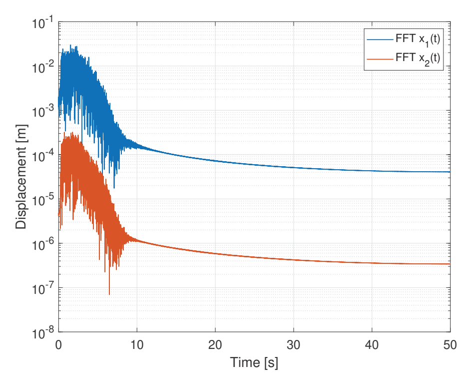
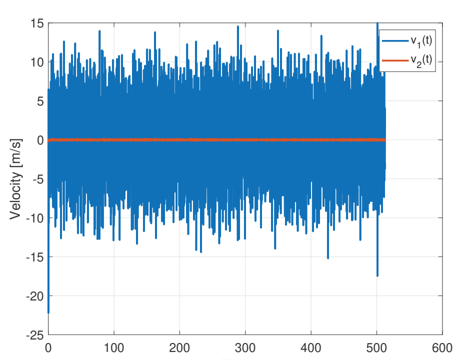
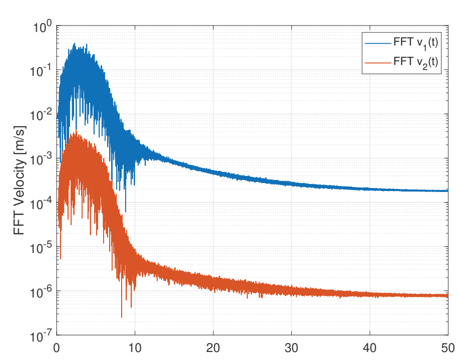
Tab. 21: SCENARIO A203: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A203	121	0.500	0.010	-	5	0.3	6.28	0.3	6.28

<table>
<tr><th>Method</th><th>Linear FRF estimate</th><th>Nonlin. param. estimates (rel. error)</th></tr>
<tr><td>$\mathbf{H}_1$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>$\mathbf{H}_2$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>$\mathbf{H}_1$-NIXO</td><td>
</td><td>$k_3 = 0.29993$ (0.02%)
$k_{3,cmplx} = 0.29993 - i0.00081$</td></tr>
<tr><td>$\mathbf{H}_2$-NIXO</td><td>
</td><td>$k_3 = 0.30104$ (0.35%)
$k_{3,cmplx} = 0.30104 - i0.00001$</td></tr>
<tr><td>$\mathbf{H}_1$-NIXO with lin. data provided</td><td>×</td><td>$k_3 = 0.29150$ (2.83%)
$k_{3,cmplx} = 0.29150 + i0.01422$</td></tr>
<tr><td>$\mathbf{H}_2$-NIXO with lin. data provided</td><td>×</td><td>$k_3 = 0.29798$ (0.67%)
$k_{3,cmplx} = 0.29798 + i0.00615$</td></tr>
</table>

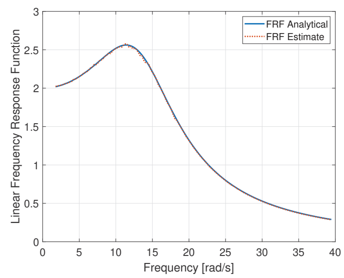
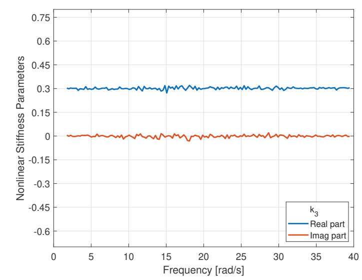
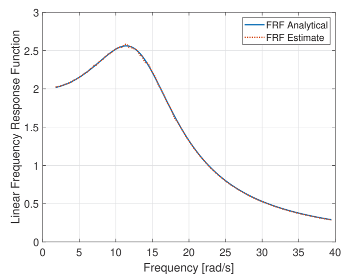
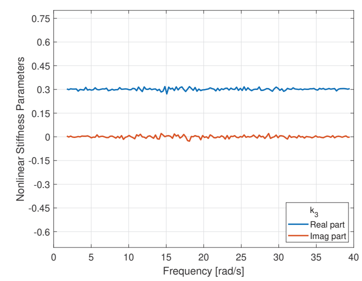
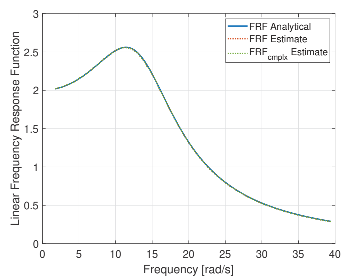
Tab. 22: SCENARIO A204: Input/Output Signals

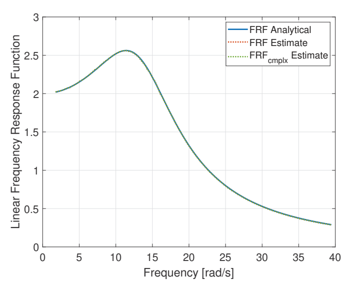
Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A204	121	1.000	0.010	-	5	0.3	6.28	0.3	6.28

Signal	Time domain	Freq. domain
$f_I(t)$		
$f_{II}(t)$		
$x(t)$		
$v(t)$		

Tab. 23: SCENARIO A204: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A204	121	1.000	0.010	-	5	0.3	6.28	0.3	6.28

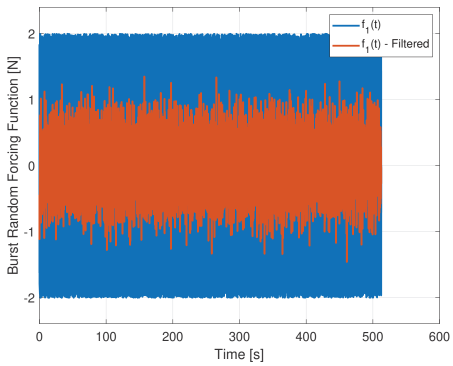
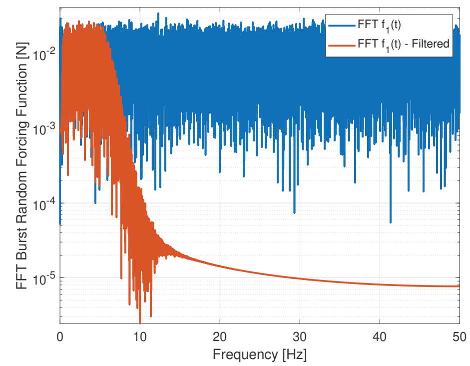
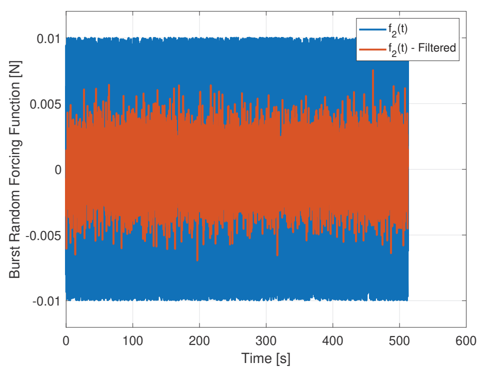
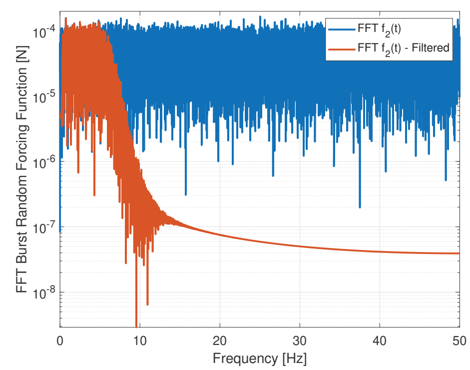
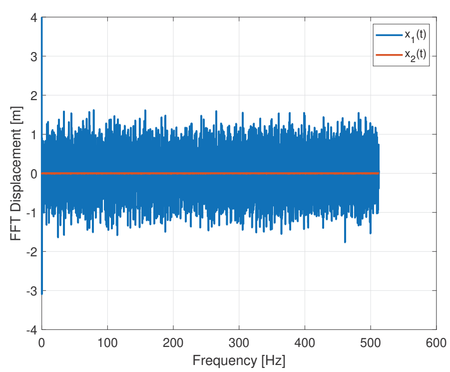
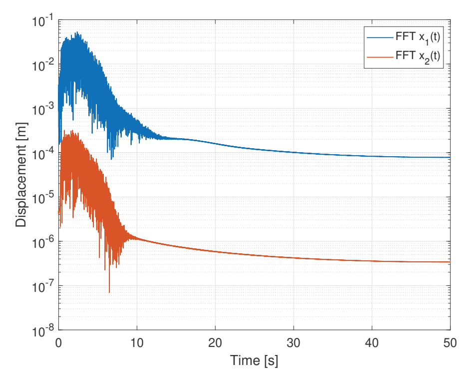
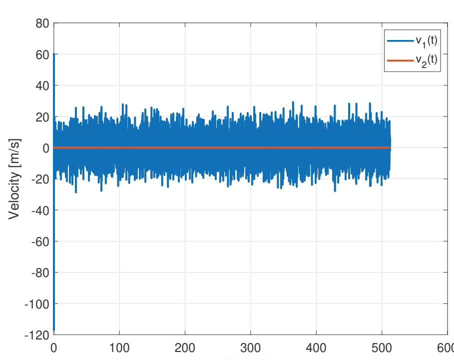
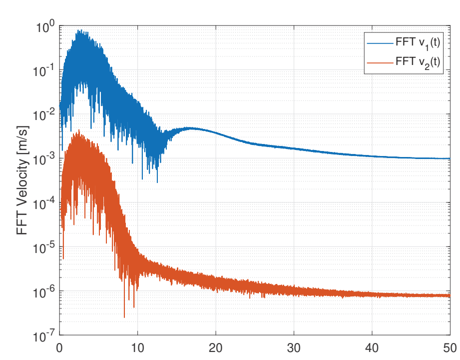
<table>
<tr><th>Method</th><th>Linear FRF estimate</th><th>Nonlin. param. estimates
(rel. error)</th></tr>
<tr><td>H$_1$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_2$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_1$-NIXO</td><td>
</td><td>$k_3 = 0.29981$ (0.06%)

$k_{3,cmplx} = 0.29981 - i0.00044$</td></tr>
<tr><td>H$_2$-NIXO</td><td>
</td><td>$k_3 = 0.30128$ (0.43%)

$k_{3,cmplx} = 0.30128 + i0.00043$</td></tr>
<tr><td>H$_1$-NIXO
with lin. data
provided</td><td>×</td><td>$k_3 = 0.29697$ (1.01%)
$k_{3,cmplx} = 0.29697 + i0.00530$</td></tr>
<tr><td>H$_2$-NIXO
with lin. data
provided</td><td>×</td><td>$k_3 = 0.29934$ (0.22%)
$k_{3,cmplx} = 0.29934 + i0.00289$</td></tr>
</table>

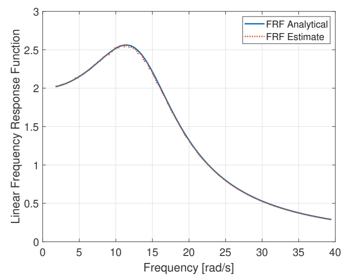
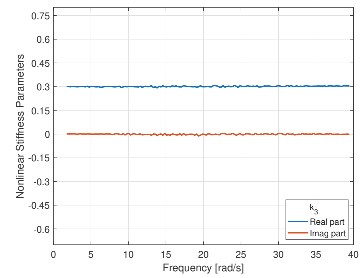
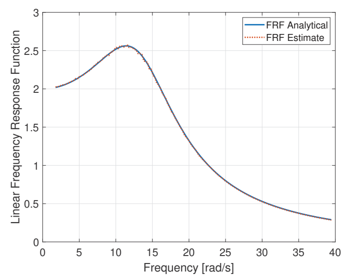
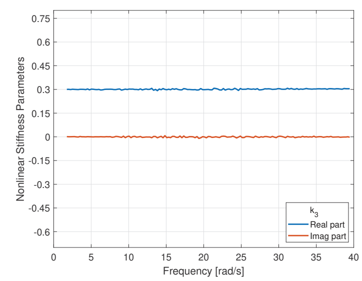
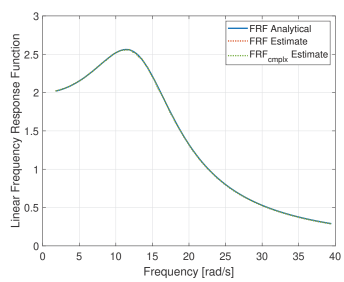
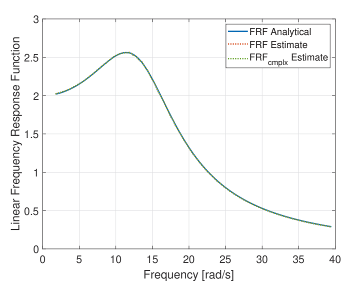
Tab. 24: SCENARIO A205: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A205	121	2.000	0.010	-	5	0.3	6.28	0.3	6.28

<table>
<tr><th>Signal</th><th>Time domain</th><th>Freq. domain</th></tr>
<tr><td>$f_I(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$f_{II}(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$x(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$v(t)$</td><td>
</td><td>
</td></tr>
</table>

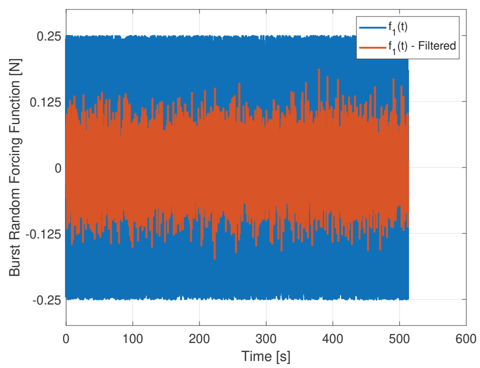
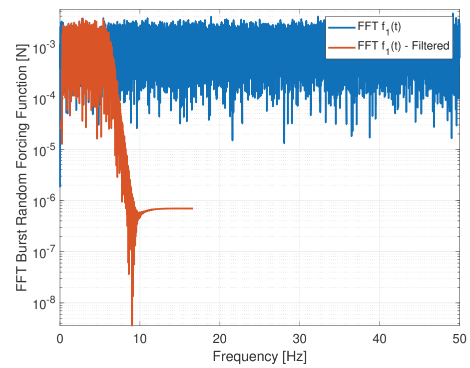
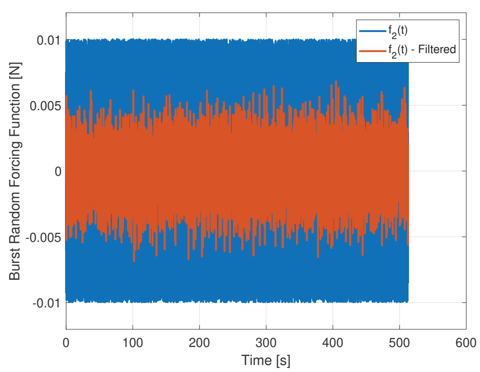
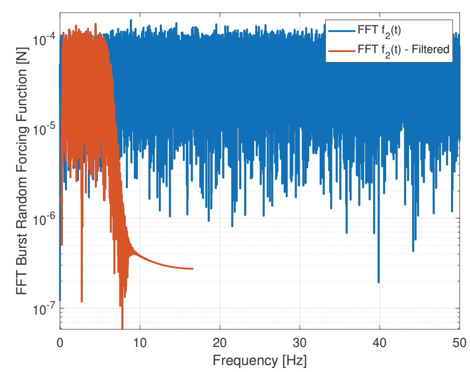
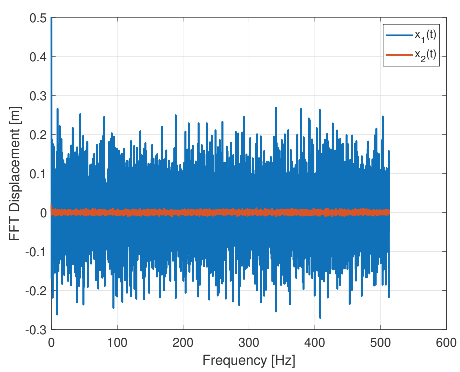
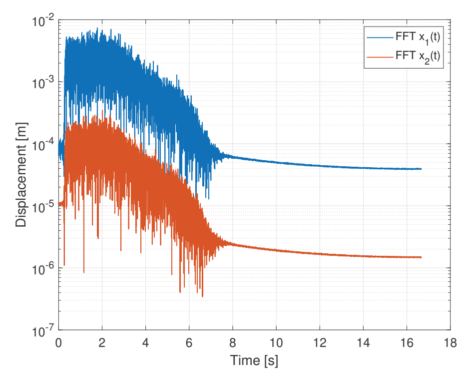
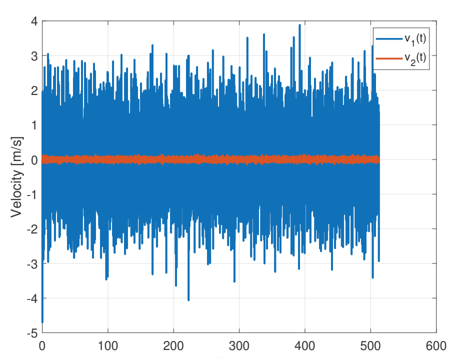
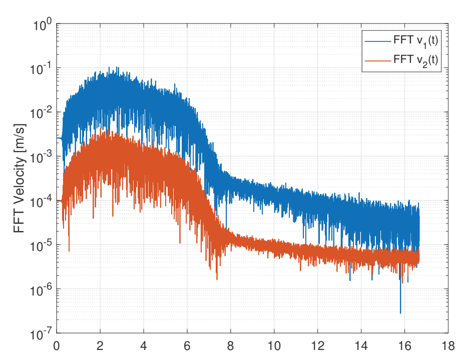
Tab. 25: SCENARIO A205: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A205	121	2.000	0.010	-	5	0.3	6.28	0.3	6.28

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
$\mathbf{H}_1$-NIFO		
$\mathbf{H}_2$-NIFO		
$\mathbf{H}_1$-NIXO		$k_3 = 0.29986$ (0.05%) $k_{3,cmplx} = 0.29986 - i0.00012$
$\mathbf{H}_2$-NIXO		$k_3 = 0.30064$ (0.21%) $k_{3,cmplx} = 0.30064 - i0.00038$
$\mathbf{H}_1$-NIXO with lin. data provided	$\times$	$k_3 = 0.29884$ (0.39%) $k_{3,cmplx} = 0.29884 + i0.00197$
$\mathbf{H}_2$-NIXO with lin. data provided	$\times$	$k_3 = 0.29939$ (0.20%) $k_{3,cmplx} = 0.29939 + i0.00115$

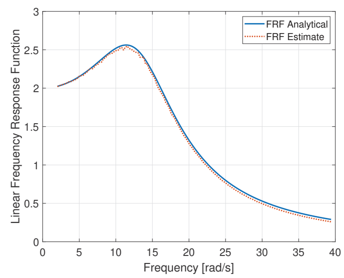
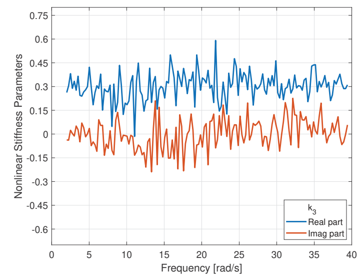
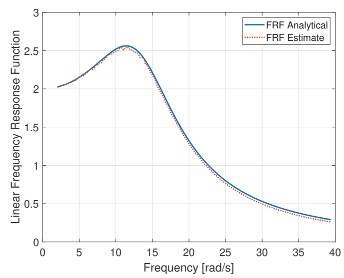
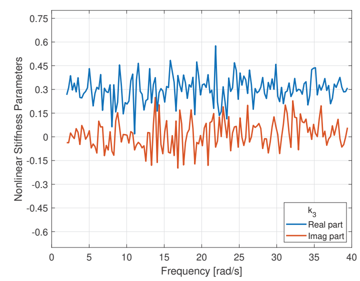
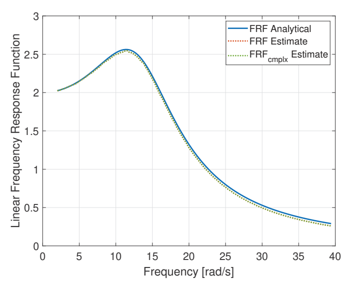
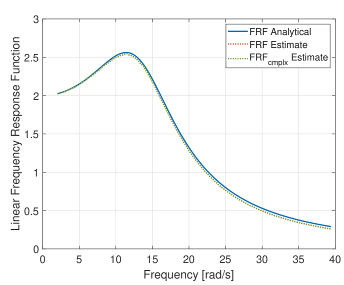
Tab. 26: SCENARIO A206: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A206	124	0.250	0.010	3	7	0.3	6.28	0.3	6.28

Signal	Time domain	Freq. domain
$f_I(t)$		
$f_{II}(t)$		
$x(t)$		
$v(t)$		

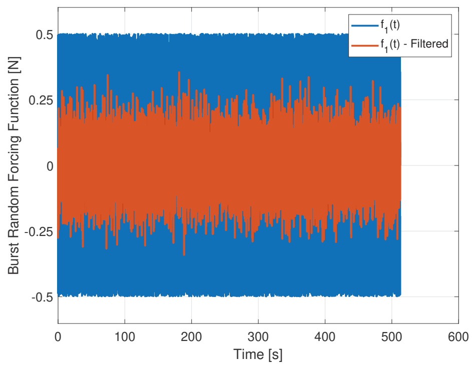
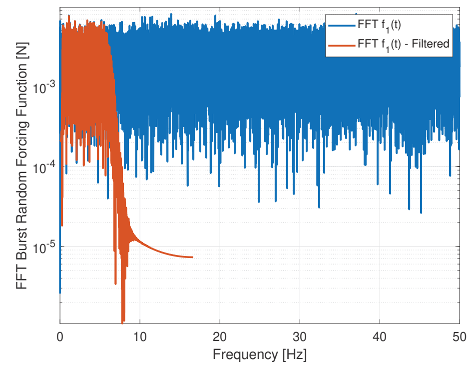
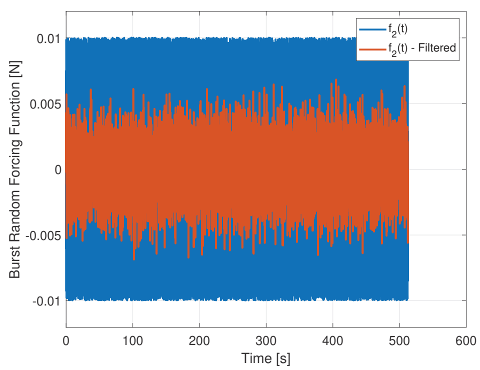
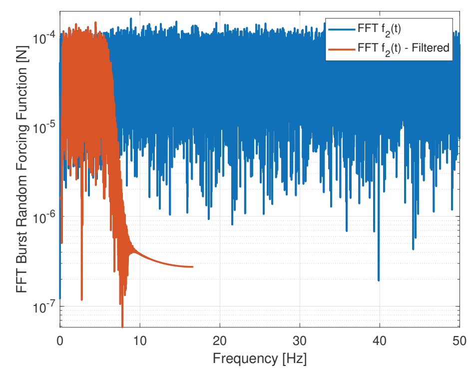
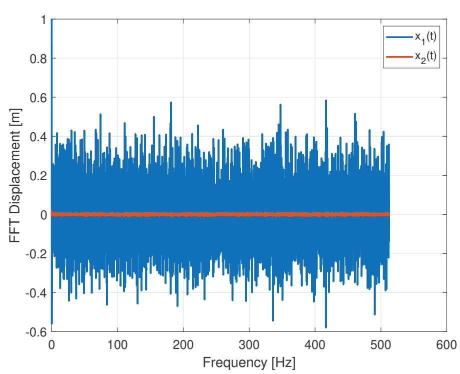
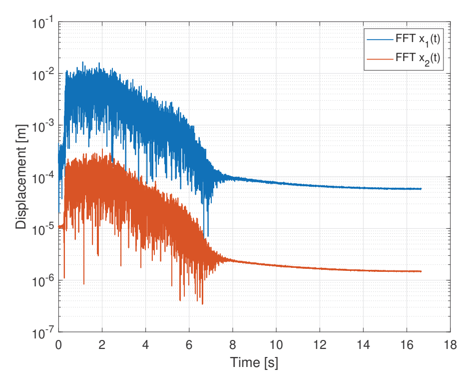
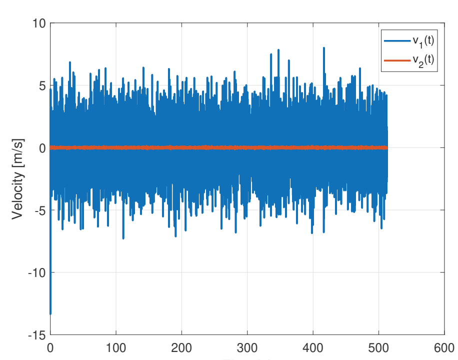
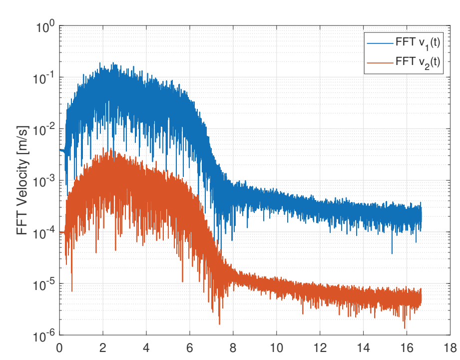
Tab. 27: SCENARIO A206: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A206	124	0.250	0.010	3	7	0.3	6.28	0.3	6.28

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
$\mathbf{H}_1$-NIFO		
$\mathbf{H}_2$-NIFO		
$\mathbf{H}_1$-NIXO		$k_3 = 0.29936$ (0.21%) $k_{3,cmplx} = 0.29936 + i0.00365$
$\mathbf{H}_2$-NIXO		$k_3 = 0.29652$ (1.16%) $k_{3,cmplx} = 0.29652 + i0.00469$
$\mathbf{H}_1$-NIXO with lin. data provided	×	$k_3 = 0.02774$ (90.75%) $k_{3,cmplx} = 0.02774 + i0.42860$
$\mathbf{H}_2$-NIXO with lin. data provided	×	$k_3 = 0.31225$ (4.08%) $k_{3,cmplx} = 0.31225 + i0.23274$

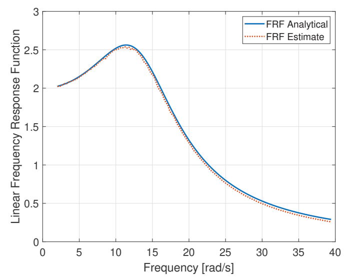
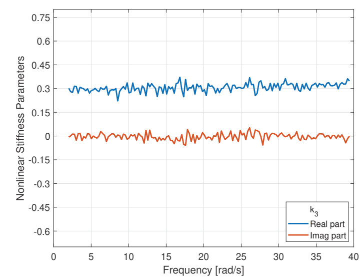
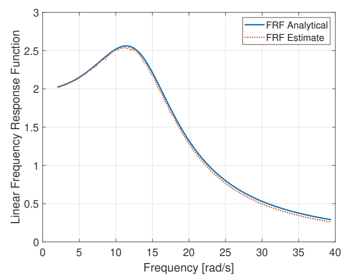
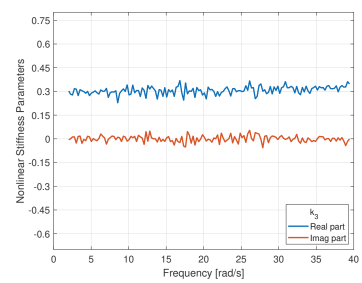
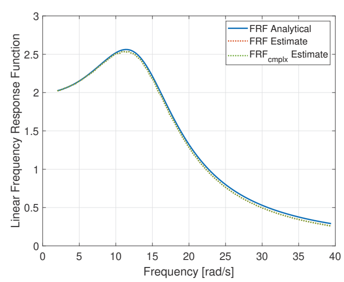
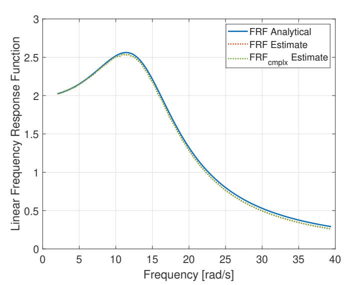
Tab. 28: SCENARIO A207: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A207	124	0.500	0.010	3	7	0.3	6.28	0.3	6.28

<table>
<tr><th>Signal</th><th>Time domain</th><th>Freq. domain</th></tr>
<tr><td>$f_I(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$f_{II}(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$x(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$v(t)$</td><td>
</td><td>
</td></tr>
</table>

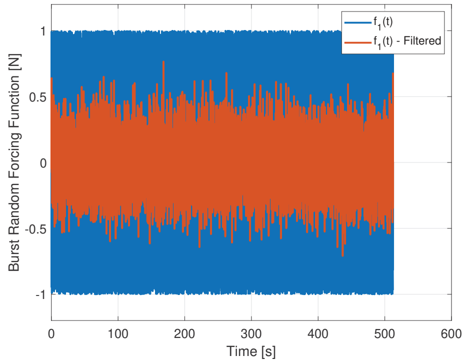
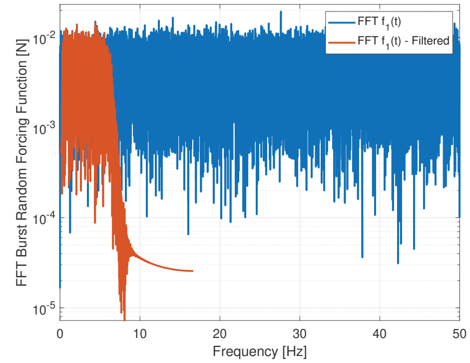
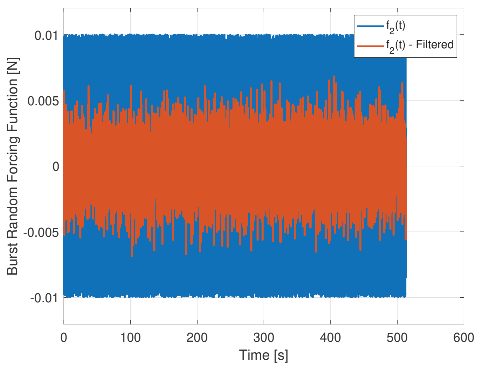
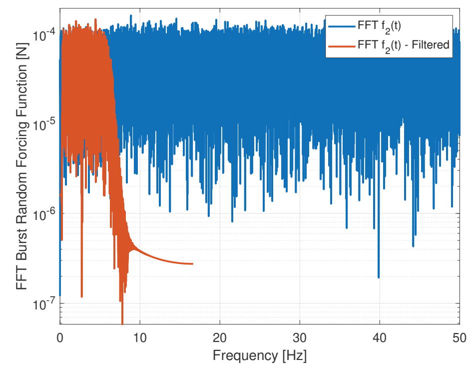
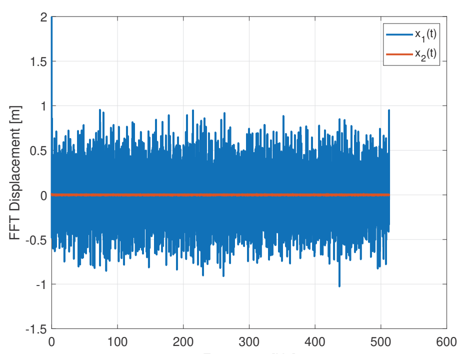
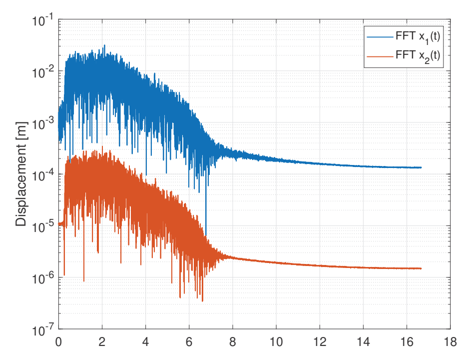
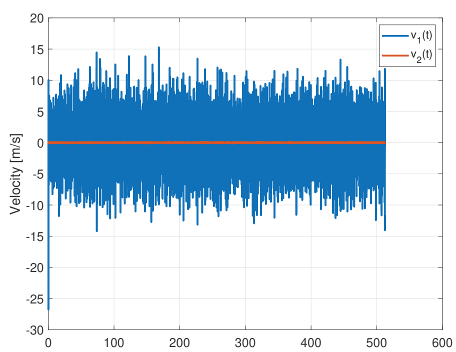
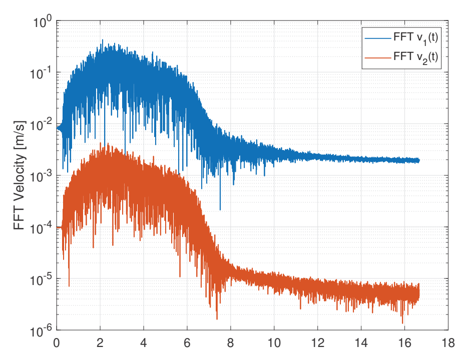
Tab. 29: SCENARIO A207: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A207	124	0.500	0.010	3	7	0.3	6.28	0.3	6.28

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 0.30220$ (0.73%) $k_{3,cmplx} = 0.30220 - i0.00105$
H$_2$-NIXO		$k_3 = 0.30097$ (0.32%) $k_{3,cmplx} = 0.30097 - i0.00001$
H$_1$-NIXO with lin. data provided	×	$k_3 = 0.23375$ (22.08%) $k_{3,cmplx} = 0.23375 + i0.10082$
H$_2$-NIXO with lin. data provided	×	$k_3 = 0.30530$ (1.77%) $k_{3,cmplx} = 0.30530 + i0.05635$

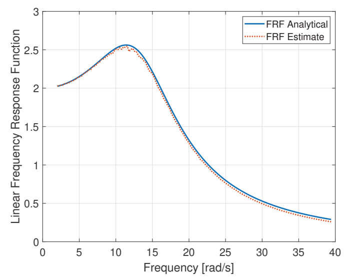
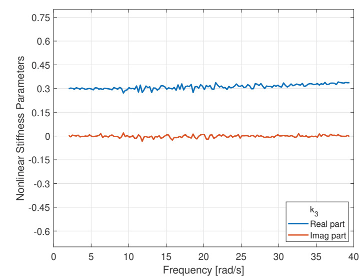
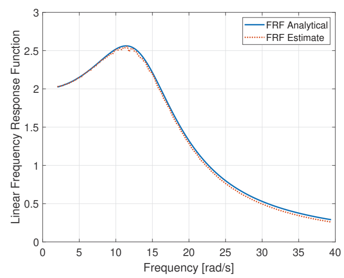
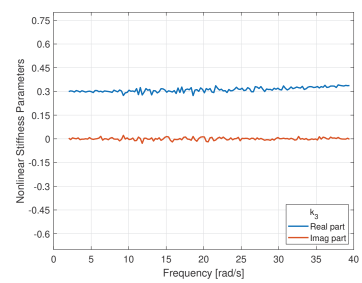
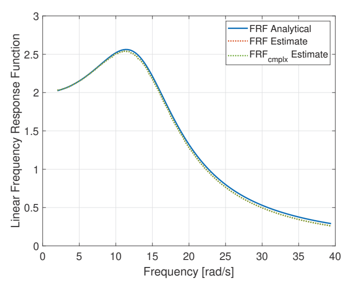
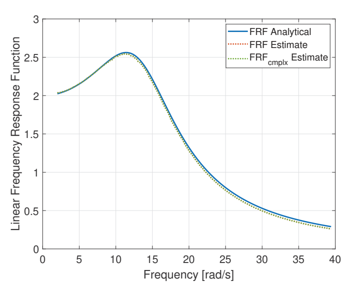
Tab. 30: SCENARIO A208: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A208	124	1.000	0.010	3	7	0.3	6.28	0.3	6.28

<table>
<tr><th>Signal</th><th>Time domain</th><th>Freq. domain</th></tr>
<tr><td>$f_I(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$f_{II}(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$x(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$v(t)$</td><td>
</td><td>
</td></tr>
</table>

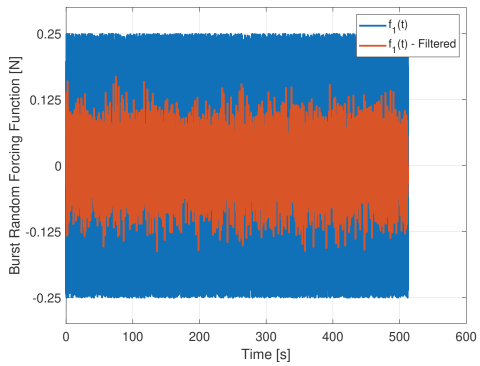
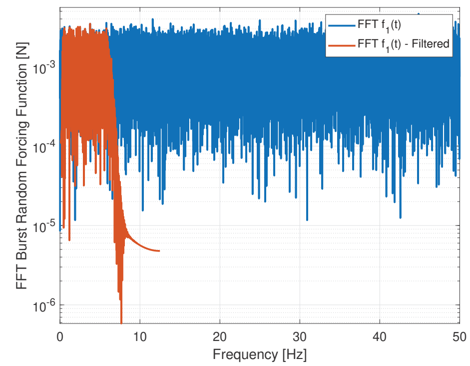
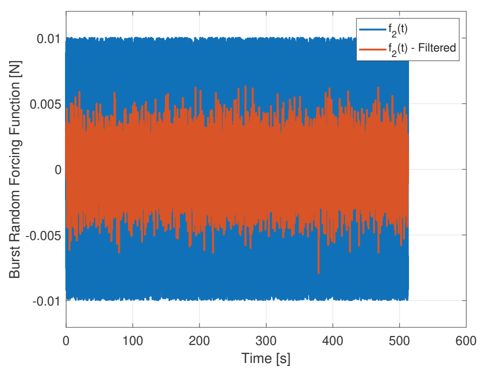
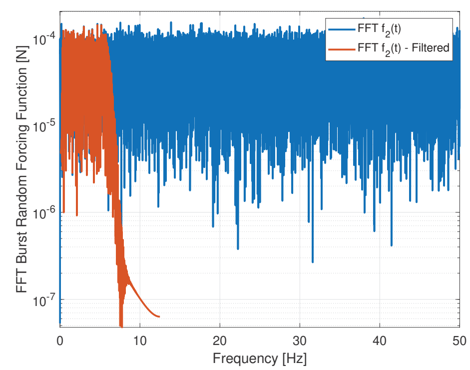
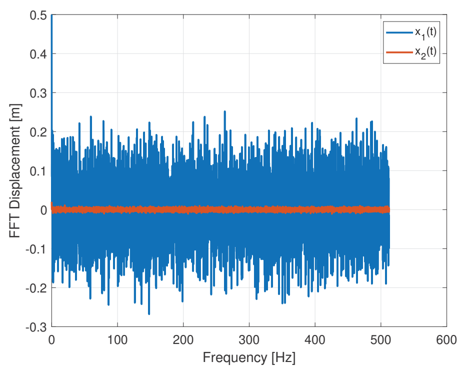
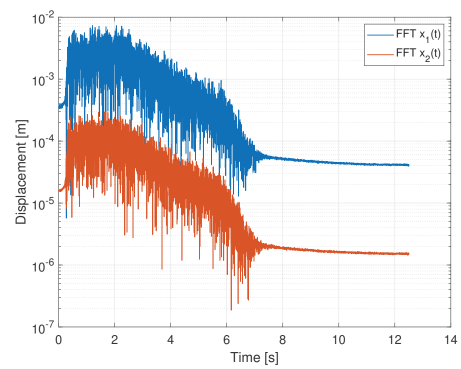
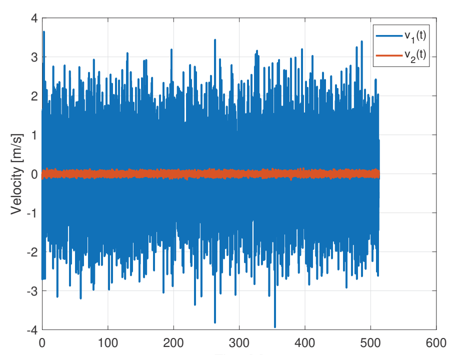
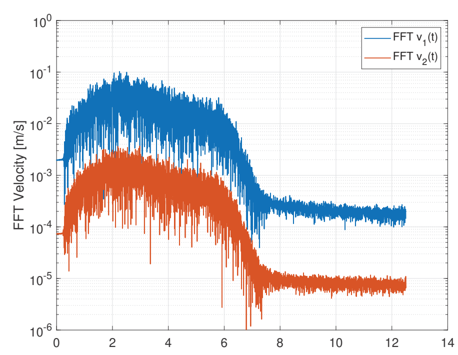
Tab. 31: SCENARIO A208: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A208	124	1.000	0.010	3	7	0.3	6.28	0.3	6.28

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 0.30244$ (0.81%) $k_{3,cmplx} = 0.30244 \; - \; i0.00010$
H$_2$-NIXO		$k_3 = 0.30660$ (2.20%) $k_{3,cmplx} = 0.30660 \; - \; i0.00029$
H$_1$-NIXO with lin. data provided	×	$k_3 = 0.27957$ (6.81%) $k_{3,cmplx} = 0.27957 \; + \; i0.03562$
H$_2$-NIXO with lin. data provided	×	$k_3 = 0.29968$ (0.11%) $k_{3,cmplx} = 0.29968 \; + \; i0.02065$

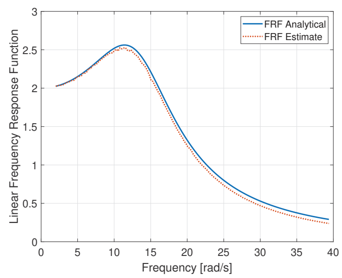
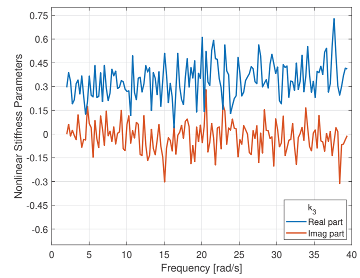
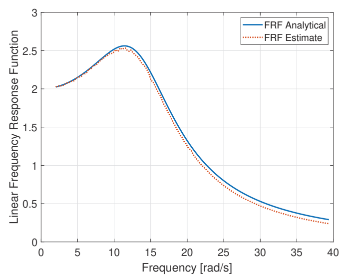
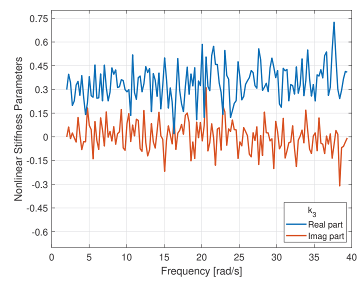
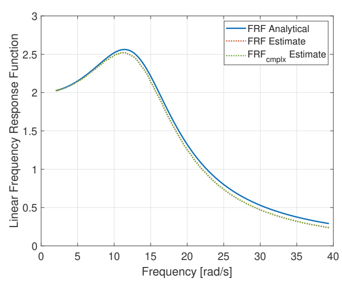
Tab. 32: SCENARIO A209: Input/Output Signals

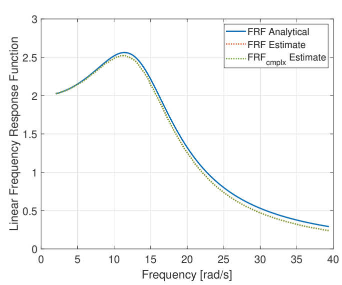
Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A209	125	0.250	0.010	4	7	0.3	6.28	0.3	6.28

Signal	Time domain	Freq. domain
$f_I(t)$		
$f_{II}(t)$		
$x(t)$		
$v(t)$		

Tab. 33: SCENARIO A209: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A209	125	0.250	0.010	4	7	0.3	6.28	0.3	6.28

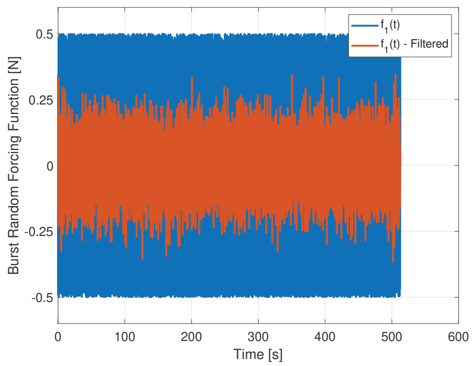
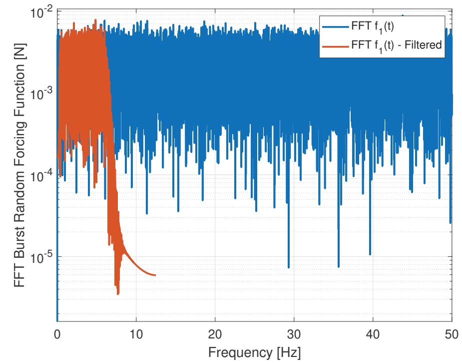
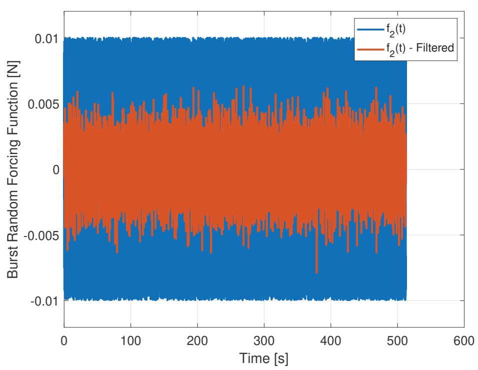
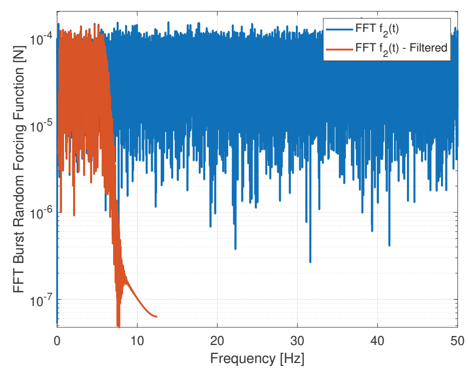
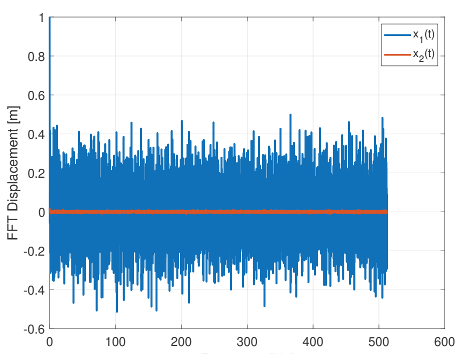
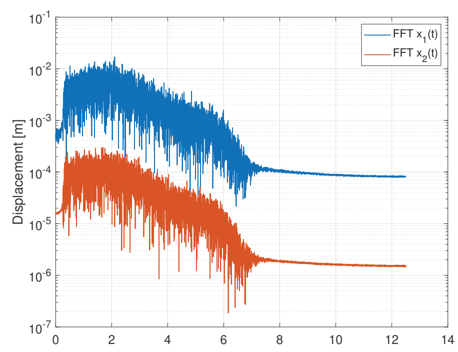
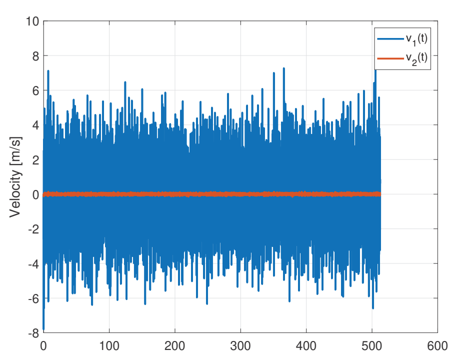
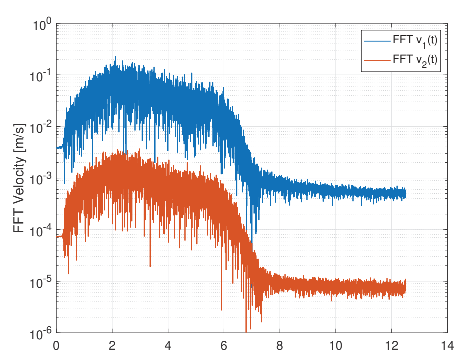
<table>
<tr><th>Method</th><th>Linear FRF estimate</th><th>Nonlin. param. estimates
(rel. error)</th></tr>
<tr><td>H$_1$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_2$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_1$-NIXO</td><td>
</td><td>$k_3 = 0.30387$ (1.29%)

$k_{3,cmplx} = 0.30387 + i0.00423$</td></tr>
<tr><td>H$_2$-NIXO</td><td>
</td><td>$k_3 = 0.32483$ (8.28%)

$k_{3,cmplx} = 0.32483 + i0.01076$</td></tr>
<tr><td>H$_1$-NIXO
with lin. data
provided</td><td>×</td><td>$k_3 = -0.19918$ (166.39%)
$k_{3,cmplx} = -0.19918 + i0.77779$</td></tr>
<tr><td>H$_2$-NIXO
with lin. data
provided</td><td>×</td><td>$k_3 = 0.30896$ (2.99%)
$k_{3,cmplx} = 0.30896 + i0.42199$</td></tr>
</table>

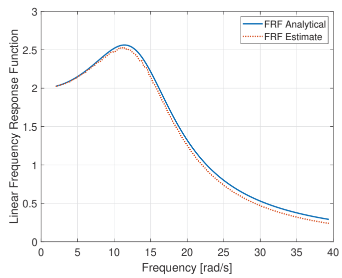
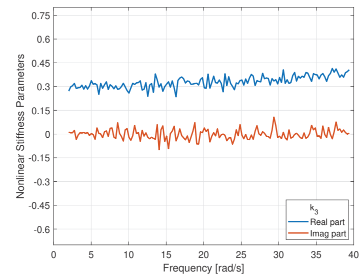
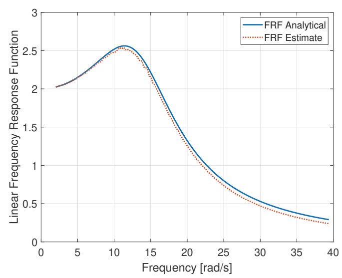
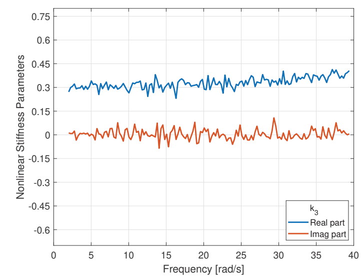
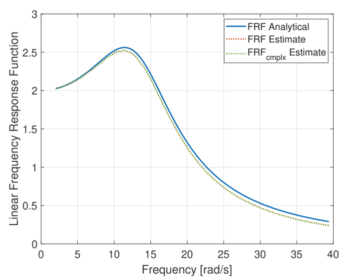
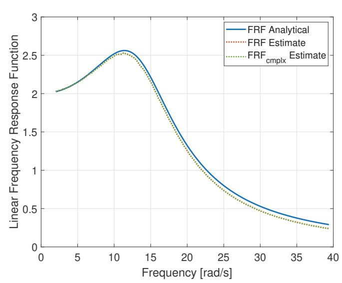
Tab. 34: SCENARIO A210: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A210	125	0.500	0.010	4	7	0.3	6.28	0.3	6.28

Signal	Time domain	Freq. domain
$f_I(t)$		
$f_{II}(t)$		
$x(t)$		
$v(t)$		

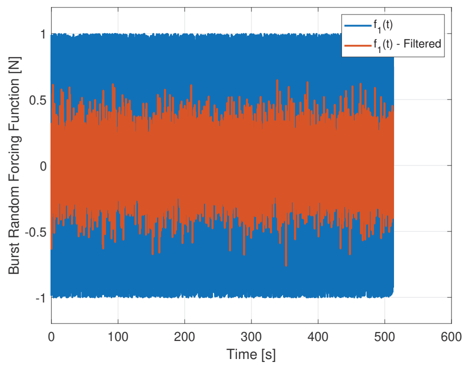
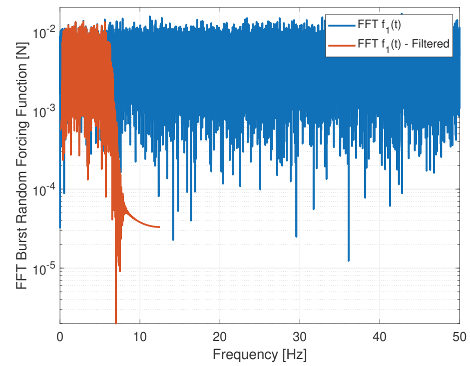
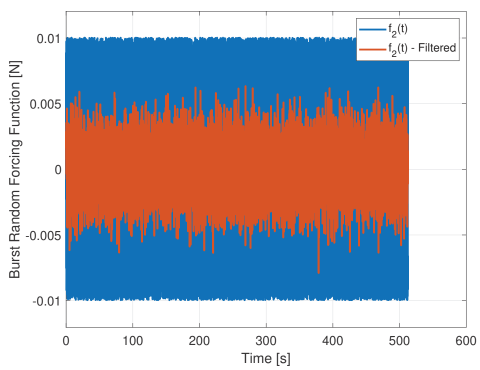
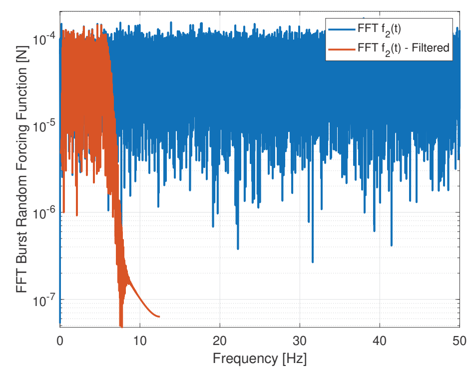
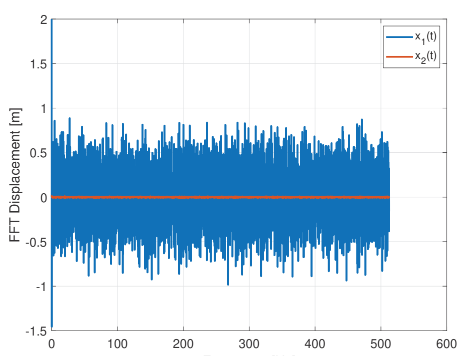
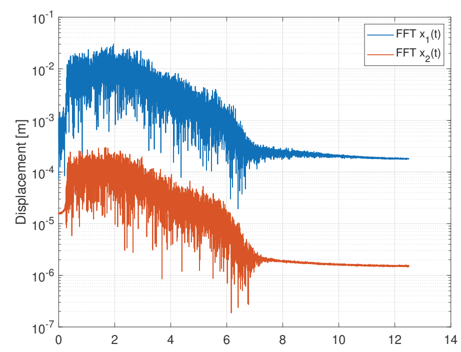
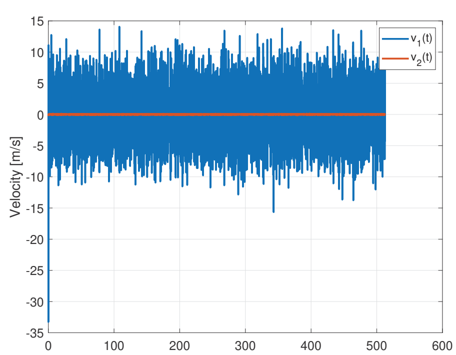
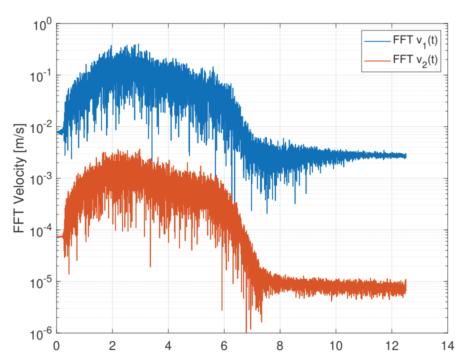
Tab. 35: SCENARIO A210: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A210	125	0.500	0.010	4	7	0.3	6.28	0.3	6.28

<table>
<tr><th>Method</th><th>Linear FRF estimate</th><th>Nonlin. param. estimates (rel. error)</th></tr>
<tr><td>H$_1$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_2$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_1$-NIXO</td><td>
</td><td>$k_3 = 0.30751$ (2.50%)
$k_{3,cmplx} = 0.30751 + i0.00097$</td></tr>
<tr><td>H$_2$-NIXO</td><td>
</td><td>$k_3 = 0.31642$ (5.47%)
$k_{3,cmplx} = 0.31642 + i0.00435$</td></tr>
<tr><td>H$_1$-NIXO with lin. data provided</td><td>×</td><td>$k_3 = 0.15736$ (47.55%)
$k_{3,cmplx} = 0.15736 + i0.20705$</td></tr>
<tr><td>H$_2$-NIXO with lin. data provided</td><td>×</td><td>$k_3 = 0.30969$ (3.23%)
$k_{3,cmplx} = 0.30969 + i0.11746$</td></tr>
</table>

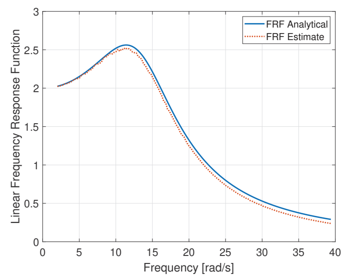
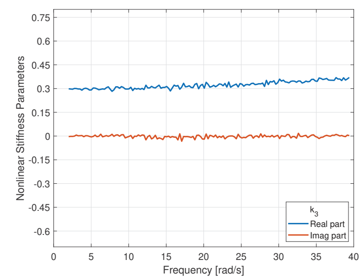
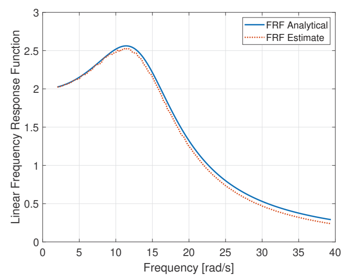
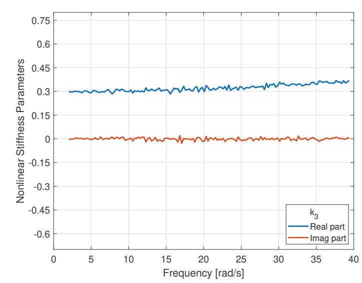
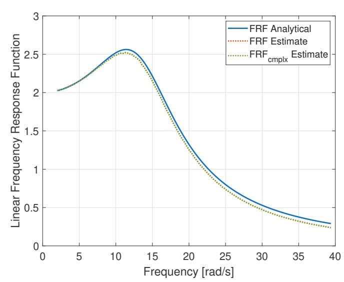
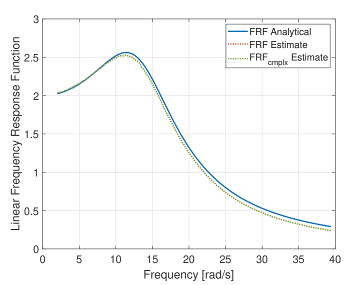
Tab. 36: SCENARIO A211: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A211	125	1.000	0.010	4	7	0.3	6.28	0.3	6.28

<table>
<tr><th>Signal</th><th>Time domain</th><th>Freq. domain</th></tr>
<tr><td>$f_I(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$f_{II}(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$x(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$v(t)$</td><td>
</td><td>
</td></tr>
</table>

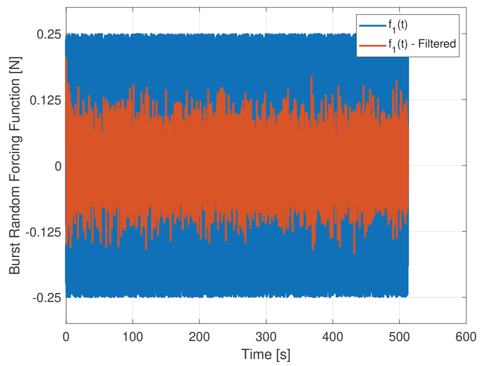
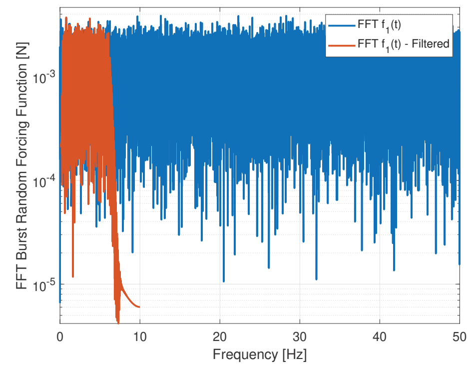
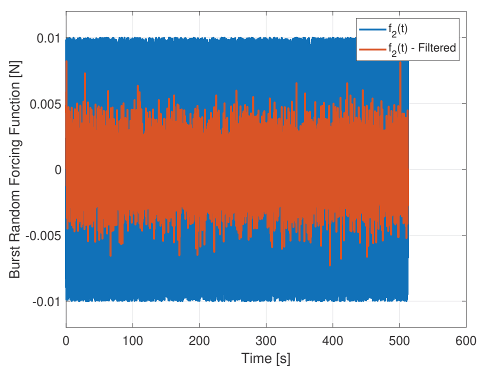
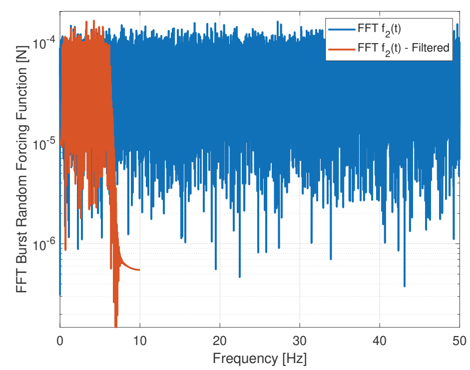
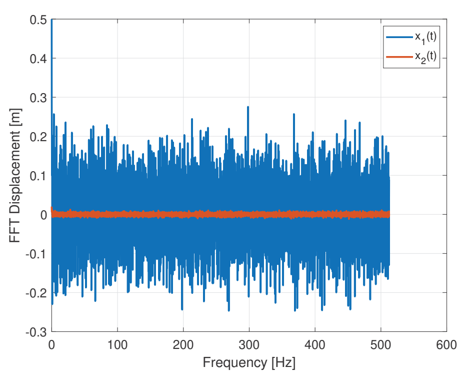
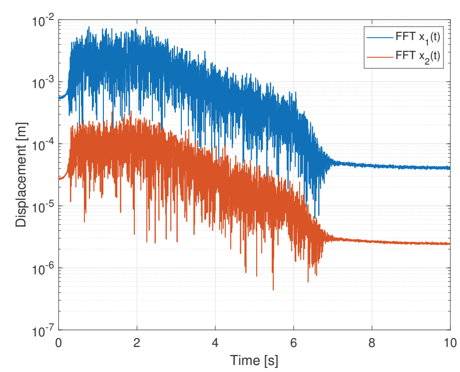
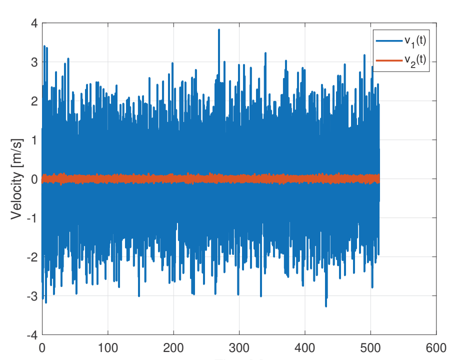
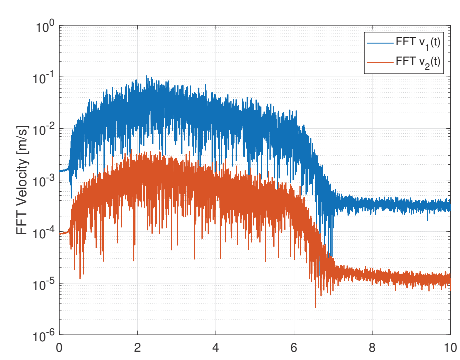
Tab. 37: SCENARIO A211: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A211	125	1.000	0.010	4	7	0.3	6.28	0.3	6.28

<table>
<tr><th>Method</th><th>Linear FRF estimate</th><th>Nonlin. param. estimates (rel. error)</th></tr>
<tr><td>H$_1$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_2$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_1$-NIXO</td><td>
</td><td>$k_3 = 0.30580$ (1.93%)
$k_{3,cmplx} = 0.30580 + i0.00029$</td></tr>
<tr><td>H$_2$-NIXO</td><td>
</td><td>$k_3 = 0.31317$ (4.39%)
$k_{3,cmplx} = 0.31317 - i0.00077$</td></tr>
<tr><td>H$_1$-NIXO with lin. data provided</td><td>×</td><td>$k_3 = 0.26379$ (12.07%)
$k_{3,cmplx} = 0.26379 + i0.06222$</td></tr>
<tr><td>H$_2$-NIXO with lin. data provided</td><td>×</td><td>$k_3 = 0.29996$ (0.01%)
$k_{3,cmplx} = 0.29996 + i0.03778$</td></tr>
</table>

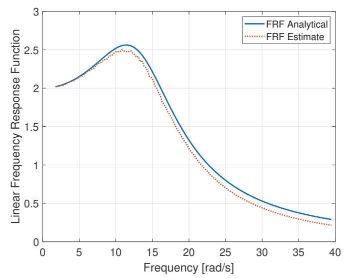
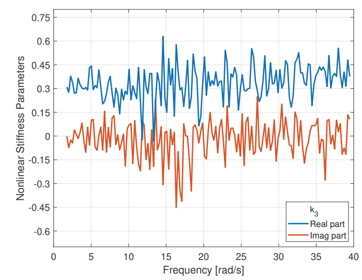
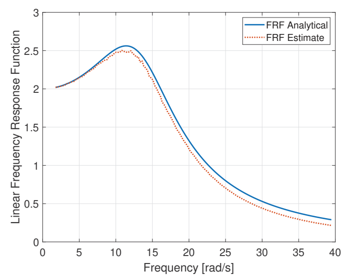
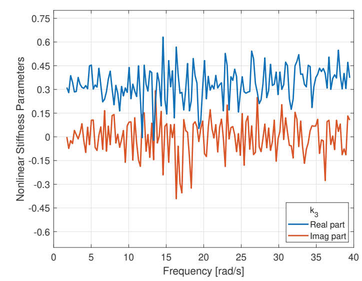
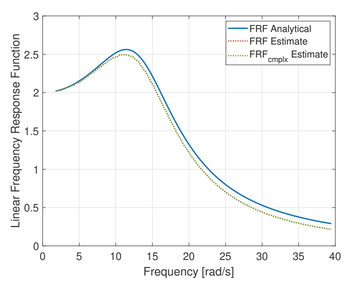
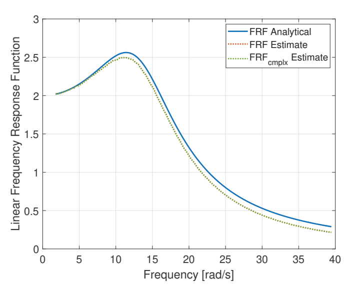
Tab. 38: SCENARIO A212: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A212	126	0.250	0.010	5	7	0.3	6.28	0.3	6.28

Signal	Time domain	Freq. domain
$f_I(t)$		
$f_{II}(t)$		
$x(t)$		
$v(t)$		

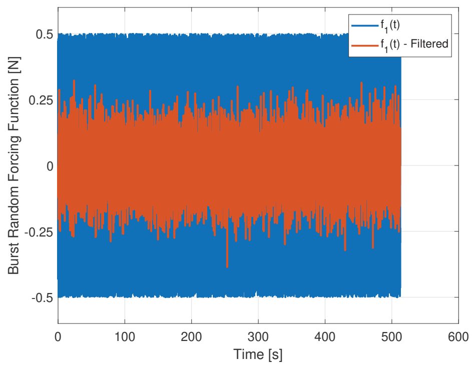
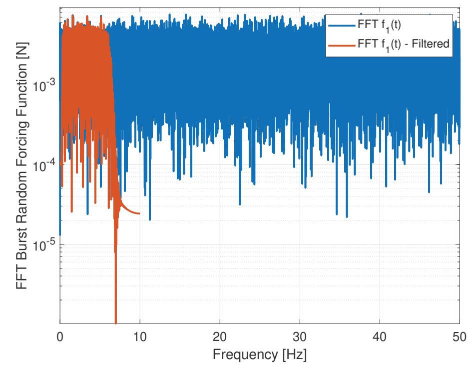
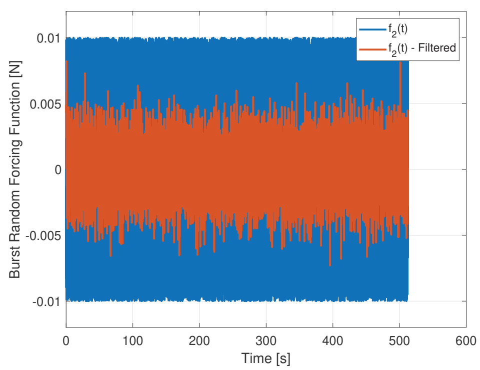
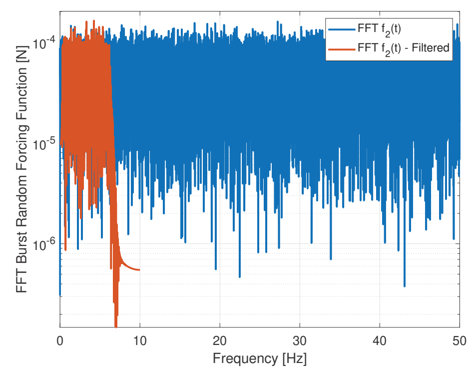
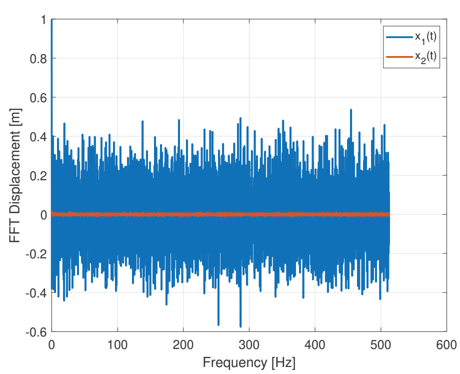
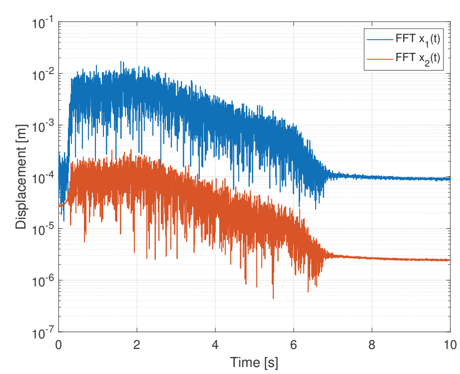
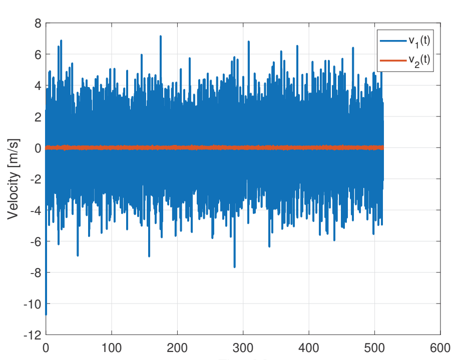
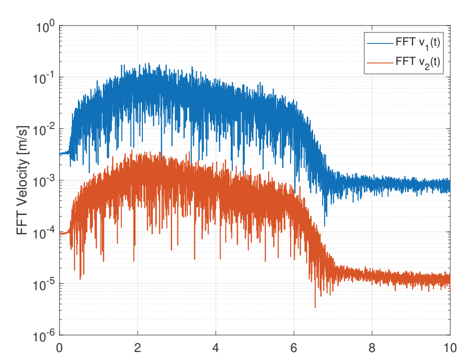
Tab. 39: SCENARIO A212: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A212	126	0.250	0.010	5	7	0.3	6.28	0.3	6.28

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 0.31538$ (5.13%) $k_{3,cmplx} = 0.31538 \;-\; i0.00561$
H$_2$-NIXO		$k_3 = 0.31769$ (5.90%) $k_{3,cmplx} = 0.31769 \;-\; i0.00519$
H$_1$-NIXO with lin. data provided	×	$k_3 = -0.27075$ (190.25%) $k_{3,cmplx} = -0.27075 \;+\; i1.00214$
H$_2$-NIXO with lin. data provided	×	$k_3 = 0.37429$ (24.76%) $k_{3,cmplx} = 0.37429 \;+\; i0.57556$

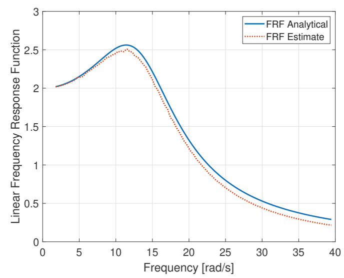
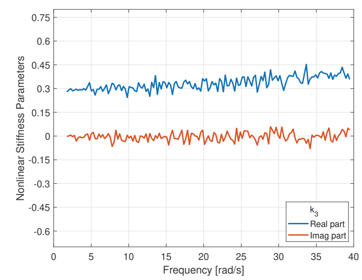
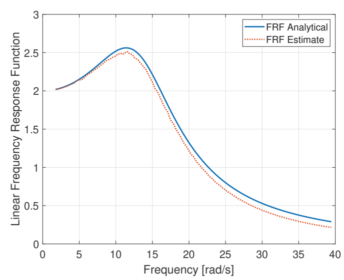
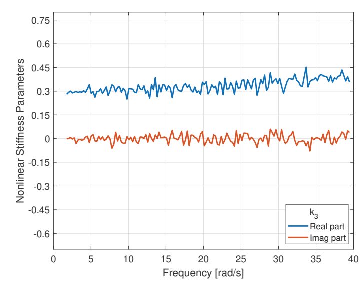
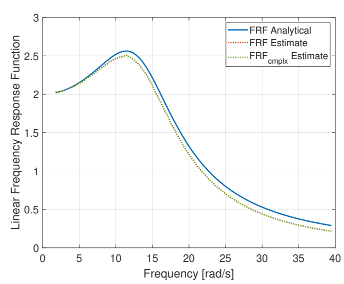
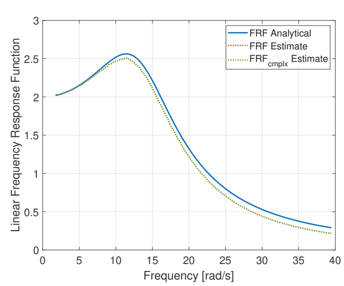
Tab. 40: SCENARIO A213: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A213	126	0.500	0.010	5	7	0.3	6.28	0.3	6.28

<table>
<tr><th>Signal</th><th>Time domain</th><th>Freq. domain</th></tr>
<tr><td>$f_I(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$f_{II}(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$x(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$v(t)$</td><td>
</td><td>
</td></tr>
</table>

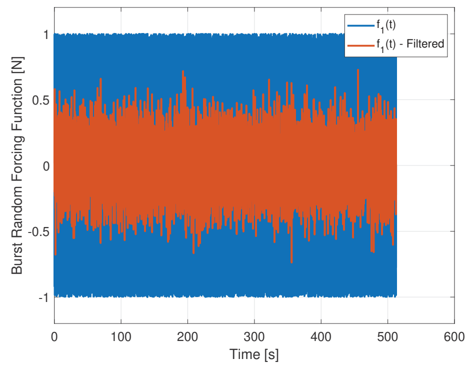
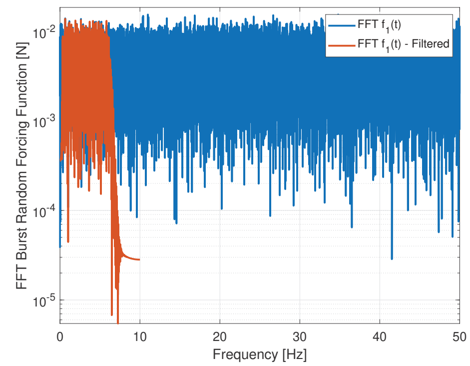
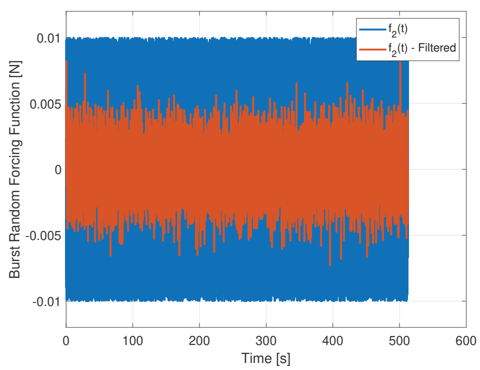
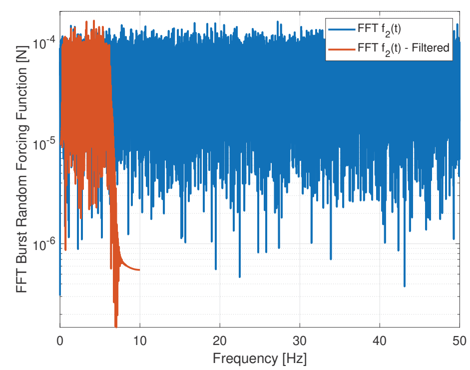
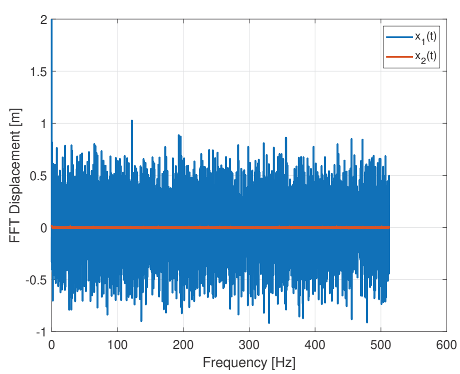
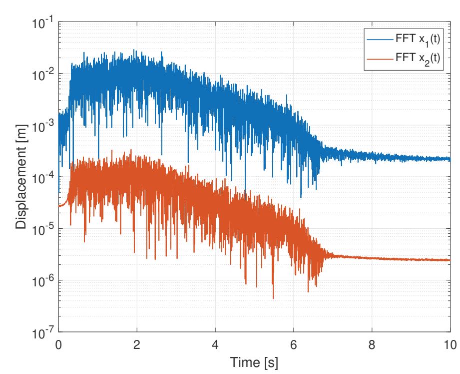
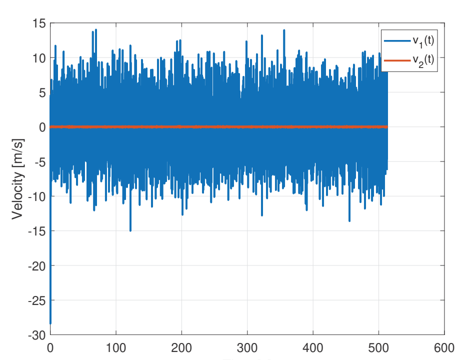
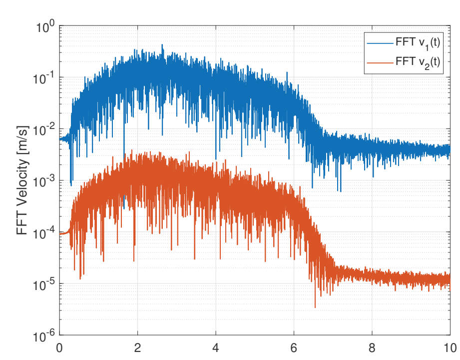
Tab. 41: SCENARIO A213: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A213	126	0.500	0.010	5	7	0.3	6.28	0.3	6.28

<table>
<tr><th>Method</th><th>Linear FRF estimate</th><th>Nonlin. param. estimates (rel. error)</th></tr>
<tr><td>H$_1$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_2$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_1$-NIXO</td><td>
</td><td>$k_3 = 0.31051$ (3.50%)
$k_{3,cmplx} = 0.31051 - i0.00086$</td></tr>
<tr><td>H$_2$-NIXO</td><td>
</td><td>$k_3 = 0.31711$ (5.70%)
$k_{3,cmplx} = 0.31711 - i0.00130$</td></tr>
<tr><td>H$_1$-NIXO with lin. data provided</td><td>×</td><td>$k_3 = 0.14067$ (53.11%)
$k_{3,cmplx} = 0.14067 + i0.27365$</td></tr>
<tr><td>H$_2$-NIXO with lin. data provided</td><td>×</td><td>$k_3 = 0.32184$ (7.28%)
$k_{3,cmplx} = 0.32184 + i0.16275$</td></tr>
</table>

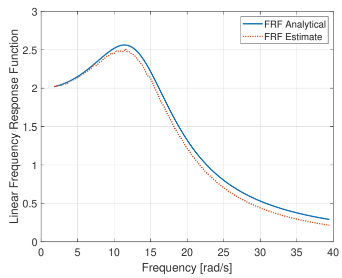
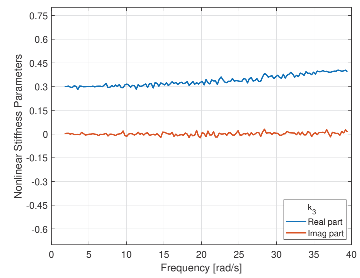
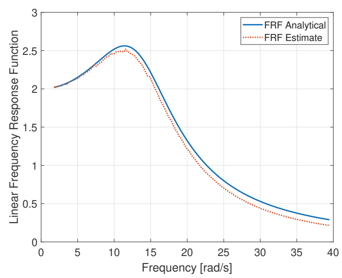
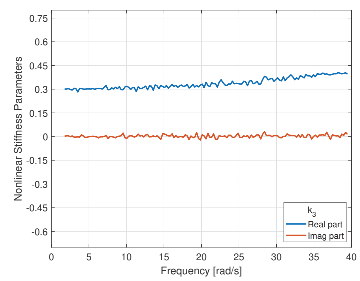
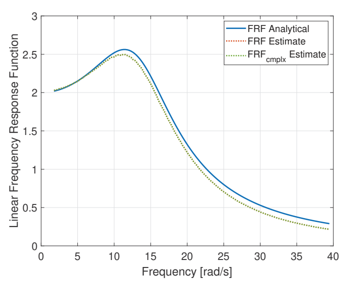
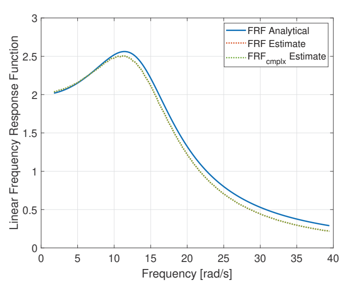
Tab. 42: SCENARIO A214: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A214	126	1.000	0.010	5	7	0.3	6.28	0.3	6.28

Signal	Time domain	Freq. domain
$f_I(t)$		
$f_{II}(t)$		
$x(t)$		
$v(t)$		

Tab. 43: SCENARIO A214: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
A214	126	1.000	0.010	5	7	0.3	6.28	0.3	6.28

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 0.31170$ (3.90%) $k_{3,cmplx} = 0.31170 \ + \ i0.00021$
H$_2$-NIXO		$k_3 = 0.32076$ (6.92%) $k_{3,cmplx} = 0.32076 \ + \ i0.00156$
H$_1$-NIXO with lin. data provided	$\times$	$k_3 = 0.24379$ (18.74%) $k_{3,cmplx} = 0.24379 \ + \ i0.09506$
H$_2$-NIXO with lin. data provided	$\times$	$k_3 = 0.30516$ (1.72%) $k_{3,cmplx} = 0.30516 \ + \ i0.05620$

Comments

1. Each method estimates the linear frequency response function with satisfactory precision. Although, if the signals provided are decimated algorithms underestimate the FRF in this particular example.

2. If the mechanical system oscillates at low amplitudes (e.g. scenario A201), only the $\mathbf{H}_1$- and $\mathbf{H}_2$-NIXO methods succeed in estimating the nonlinear parameter.

3. To increase algorithms precision, signals should be collected in the tests where structure vibrates at larger amplitudes (see scenarios A203-A205).

4. Estimates of the nonlinear parameter obtained with NIFO methods are affected by the decimation factor value. For large enough values of DF, k_3 starts being an increasing function of frequency (see results obtained with signals 125 or 126).

5. Estimates can be considered as accurate when $Im\{k_3\}$ is zero, or much lower than the real part of k_3.

2.2 Example 2

In this section the algorithms are again tested using the Duffing equation (29). Values describing the mechanical system were proposed in [1] and are given in Tab. 44. Auto- and cross-spectra were obtained by applying 25-seconds-long Hanning windows with 51% of overlapping.

Tab. 44: Parameters describing SDOF mechanical system with cubic stiffness nonlinearity. Example 2.

m [kg]	c $\left[\frac{\mathrm{N\,s}}{\mathrm{m}}\right]$	k $\left[\frac{\mathrm{N}}{\mathrm{m}}\right]$	k_3 $\left[\frac{\mathrm{N}}{\mathrm{m}^3}\right]$
1	4	10^3	10^5

Forcing signals and different testing scenarios are defined in:

– Tab. 45 and Tab. 46 (swept cosine),

– Tab. 61 and Tab. 62 (burst random).

2.2.1 Forcing Type: Swept Cosine

Tab. 45: Parameters describing swept cosine forcing function. Example 2.

F [N]	Ω_{st} [Hz]	Ω_{end} [Hz]	t_{st} [s]	t_{end} [s]
different values	0.01	15	0	1500

Tab. 46: Description of various scenarios the algorithms are tested with. Force amplitudes expressed in newtons, frequencies in hertz. Example 2.

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}	I/O Signals	Results
B101	101	0.5	0.1	-	-	-	-	0.3	15	Tab. 47	Tab. 48
B102		1.0								Tab. 49	Tab. 50
B103		2.5								Tab. 51	Tab. 52
B104		5.0								Tab. 53	Tab. 54
B105		7.5								Tab. 55	Tab. 56
B106	102	1.0	0.1	3	-	-	-	0.3	15	Tab. 57	Tab. 58
B107		2.5								Tab. 59	Tab. 60

DF – Decimation Factor

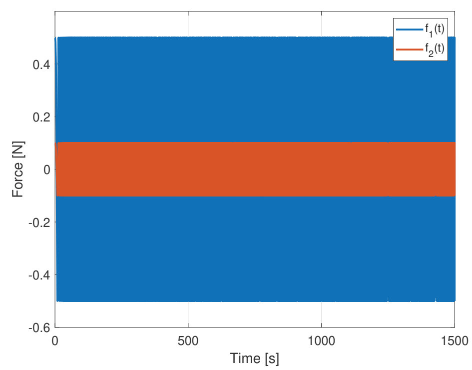
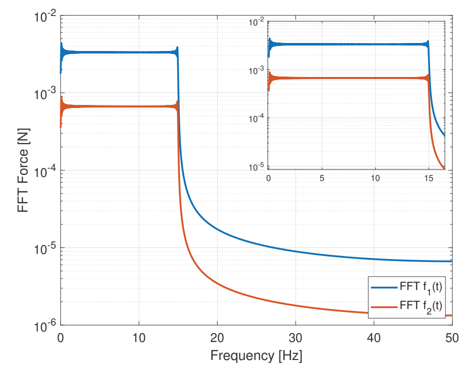
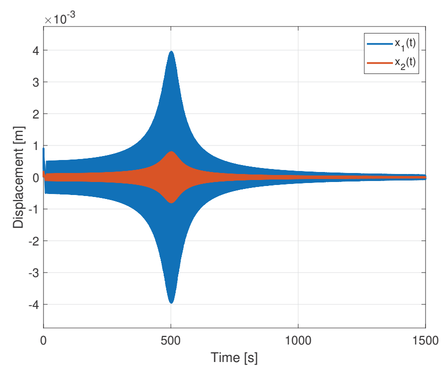
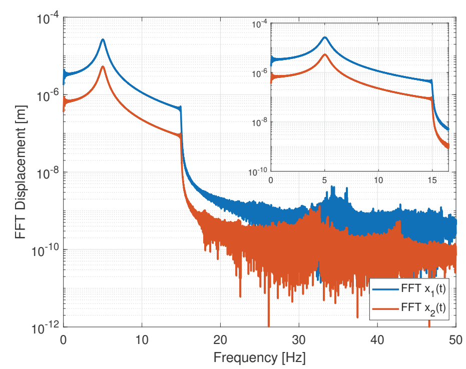
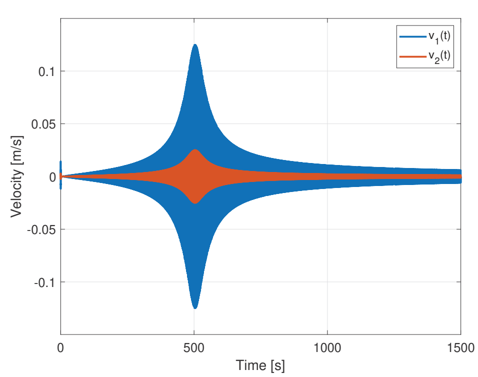
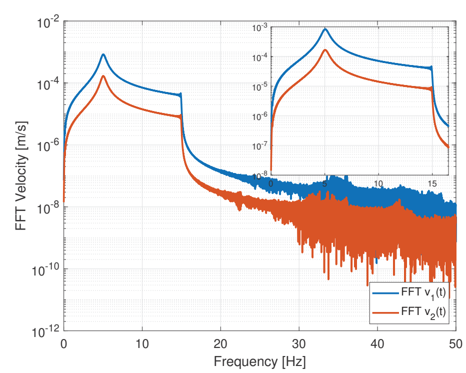
FO – Butterworth Filter Order

Ω^{filt} – cut-off frequency; Ω_1^{filt} and Ω_2^{filt} are lower and upper cut-off frequencies, respectively

Ω^{spect} – auto- and cross-spectra are computed for frequency range $(\Omega_1^{spect},\ \Omega_2^{spect})$.

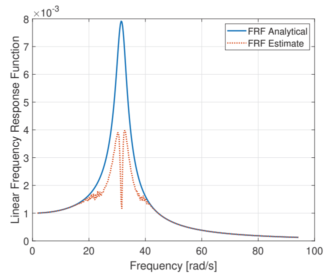
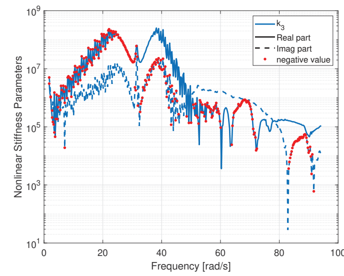
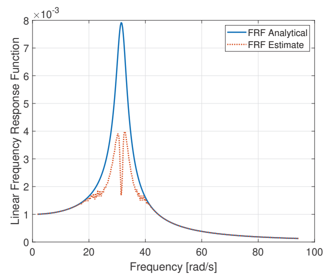
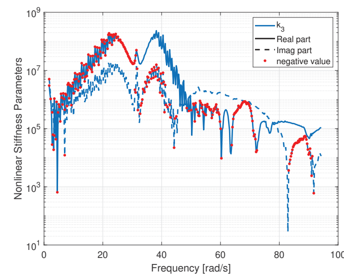
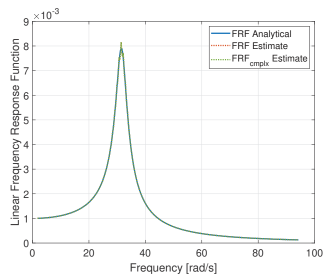
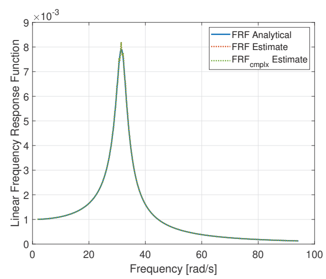
Tab. 47: SCENARIO B101: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B101	101	0.500	0.100	-	-	-	-	0.3	15

Signal	Time domain	Freq. domain
$f(t)$		
$x(t)$		
$v(t)$		

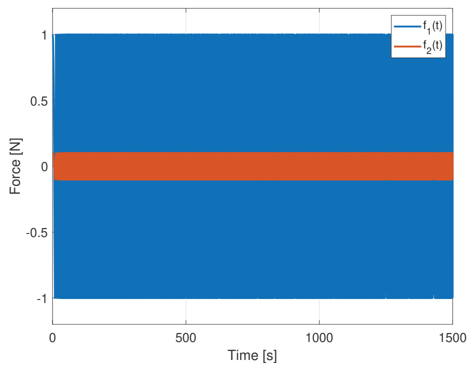
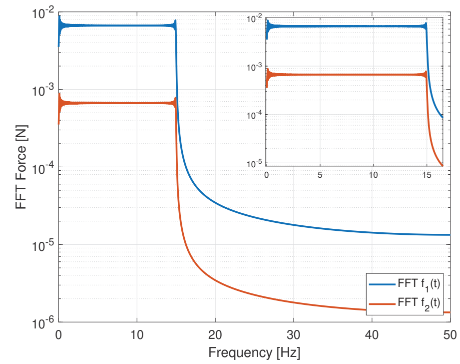
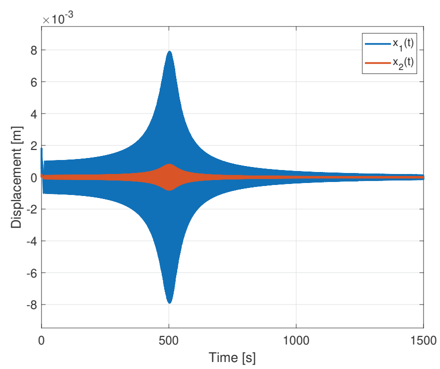
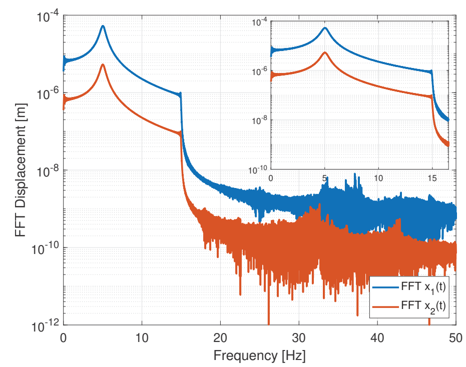
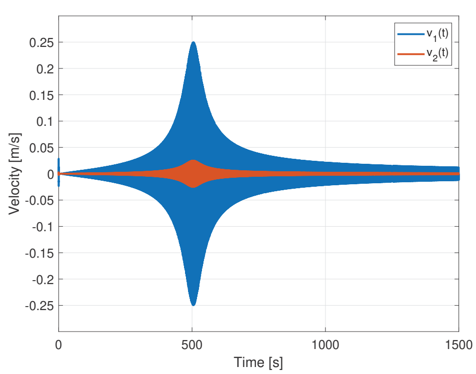
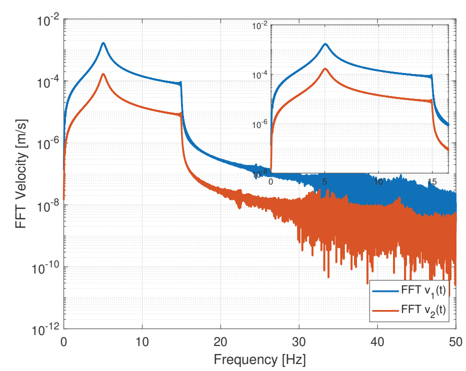
Tab. 48: SCENARIO B101: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B101	101	0.500	0.100	-	-	-	-	0.3	15

<table>
<tr><th>Method</th><th>Linear FRF estimate</th><th>Nonlin. param. estimates
(rel. error)</th></tr>
<tr><td>H$_1$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_2$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_1$-NIXO</td><td>
</td><td>$k_3 = 8.468e+04$ (15.32%)
$k_{3,cmplx} = 8.468e{+}04 - i1.534e{+}03$</td></tr>
<tr><td>H$_2$-NIXO</td><td>
</td><td>$k_3 = 9.293e+04$ (7.07%)
$k_{3,cmplx} = 9.319e{+}04 + i7.450e{+}03$</td></tr>
<tr><td>H$_1$-NIXO
with lin. data
provided</td><td>×</td><td>$k_3 = 1.291e+05$ (29.08%)
$k_{3,cmplx} = 1.291e{+}05 + i7.195e{+}04$</td></tr>
<tr><td>H$_2$-NIXO
with lin. data
provided</td><td>×</td><td>$k_3 = 1.204e+05$ (20.38%)
$k_{3,cmplx} = 1.204e{+}05 + i1.506e{+}04$</td></tr>
</table>

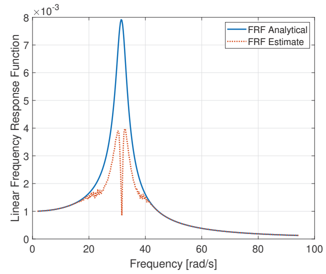
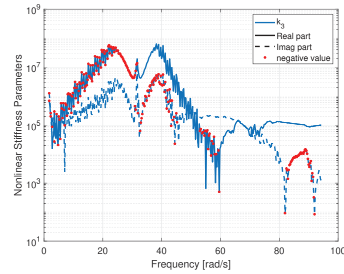
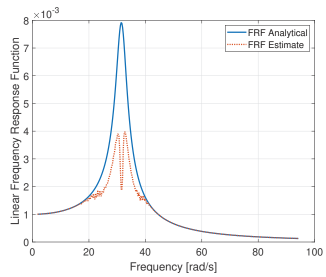
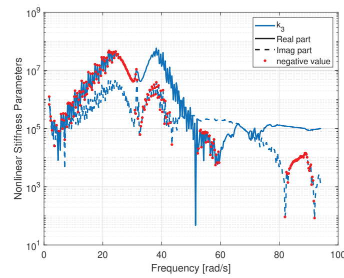
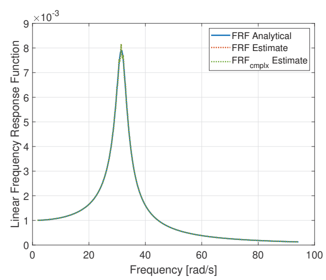
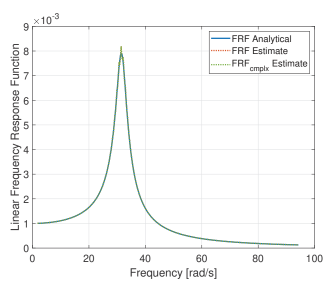
Tab. 49: SCENARIO B102: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B102	101	1.000	0.100	-	-	-	-	0.3	15

Signal	Time domain	Freq. domain
$f(t)$		
$x(t)$		
$v(t)$		

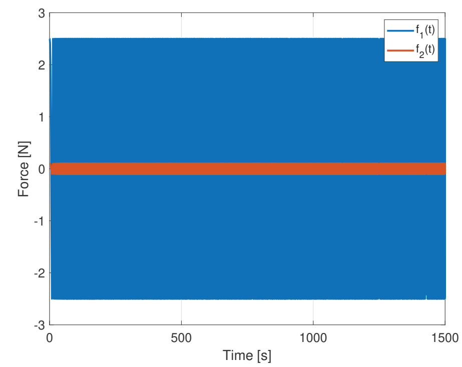
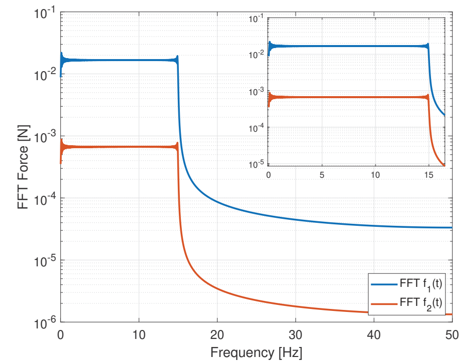
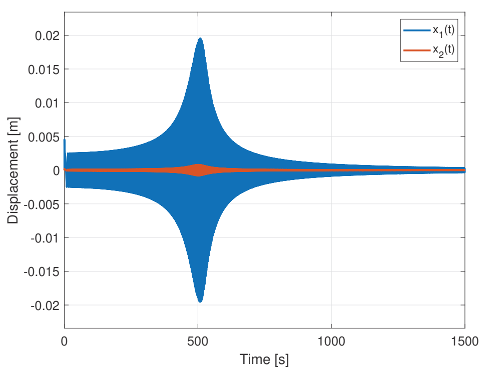
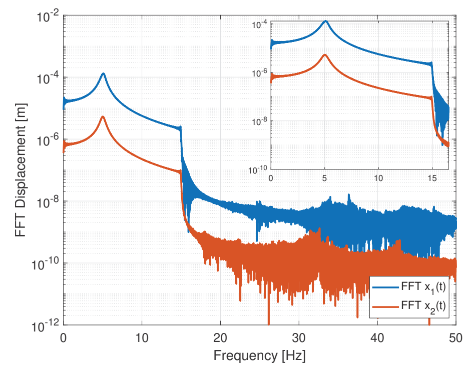
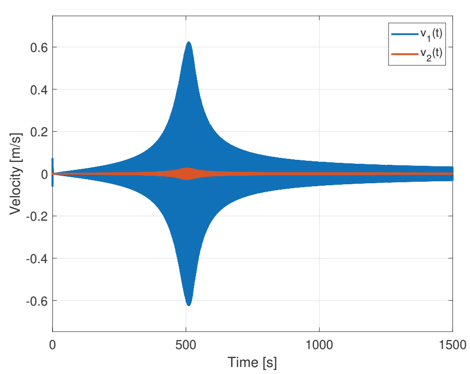
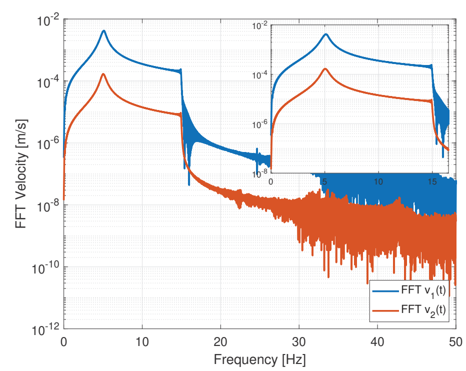
Tab. 50: SCENARIO B102: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B102	101	1.000	0.100	-	-	-	-	0.3	15

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
$\mathbf{H}_1$-NIFO		
$\mathbf{H}_2$-NIFO		
$\mathbf{H}_1$-NIXO		$k_3 = 9.576e+04$ (4.24%) $k_{3,cmplx} = 9.576e{+}04 - i5.010e{+}02$
$\mathbf{H}_2$-NIXO		$k_3 = 9.724e+04$ (2.76%) $k_{3,cmplx} = 9.724e{+}04 + i3.793e{+}02$
$\mathbf{H}_1$-NIXO with lin. data provided	×	$k_3 = 1.070e+05$ (6.99%) $k_{3,cmplx} = 1.070e{+}05 + i1.792e{+}04$
$\mathbf{H}_2$-NIXO with lin. data provided	×	$k_3 = 1.048e+05$ (4.84%) $k_{3,cmplx} = 1.048e{+}05 + i3.656e{+}03$

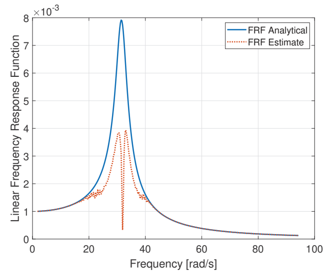
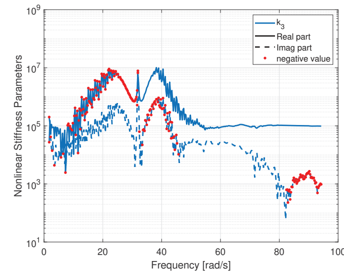
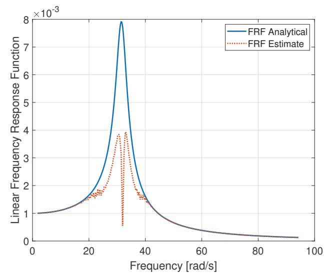
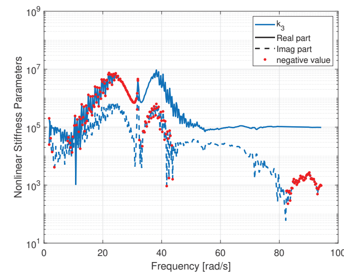
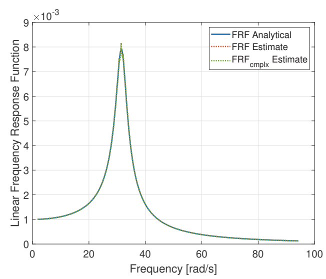
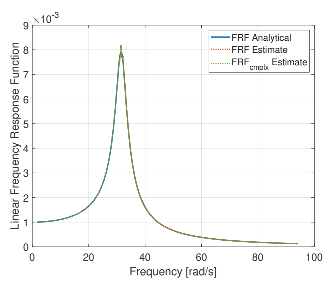
Tab. 51: SCENARIO B103: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B103	101	2.500	0.100	-	-	-	-	0.3	15

Signal	Time domain	Freq. domain
$f(t)$		
$x(t)$		
$v(t)$		

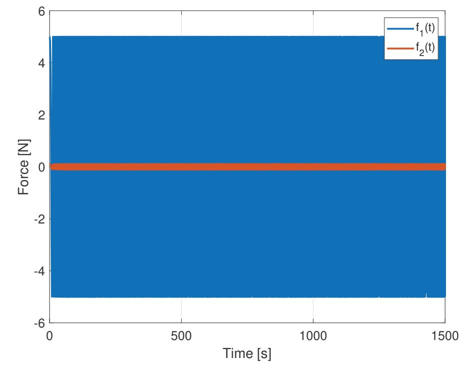
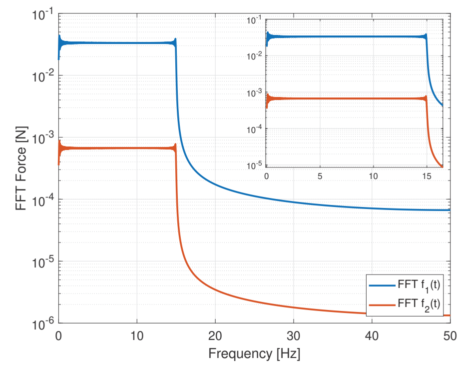
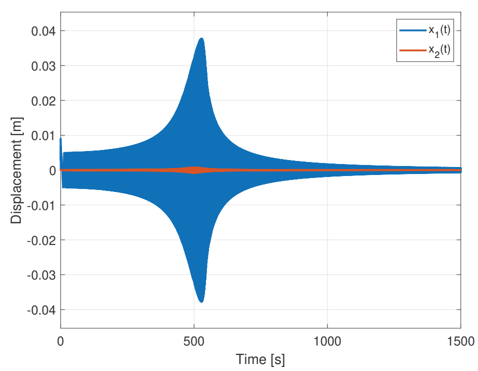
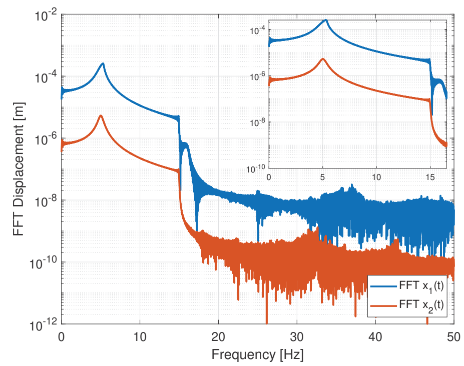
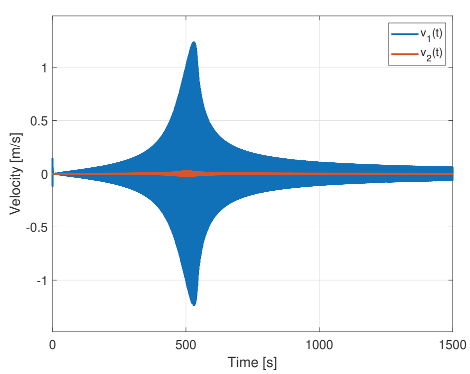
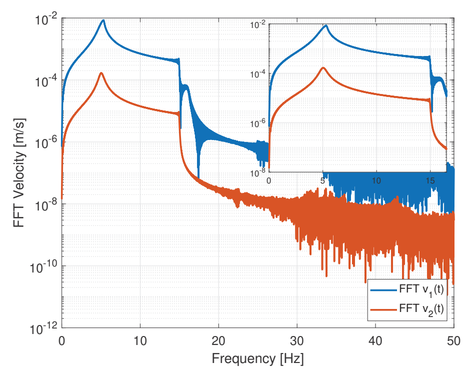
Tab. 52: SCENARIO B103: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B103	101	2.500	0.100	-	-	-	-	0.3	15

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 9.876e+04$ (1.24%) $k_{3,cmplx} = 9.876e{+}04 - i2.540e{+}02$
H$_2$-NIXO		$k_3 = 9.972e+04$ (0.28%) $k_{3,cmplx} = 9.972e{+}04 - i1.059e{+}03$
H$_1$-NIXO with lin. data provided	×	$k_3 = 1.008e+05$ (0.82%) $k_{3,cmplx} = 1.008e{+}05 + i2.780e{+}03$
H$_2$-NIXO with lin. data provided	×	$k_3 = 1.005e+05$ (0.47%) $k_{3,cmplx} = 1.005e{+}05 + i4.482e{+}02$

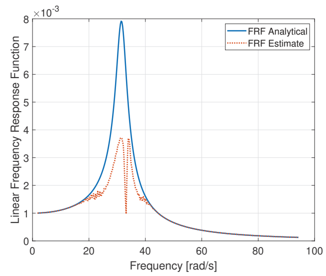
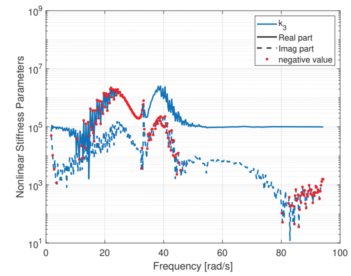
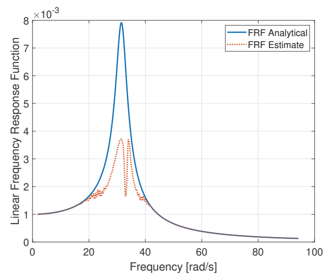
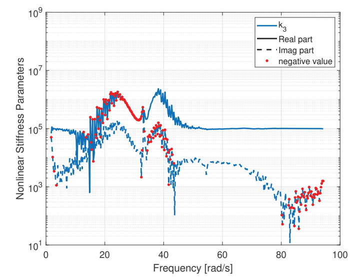
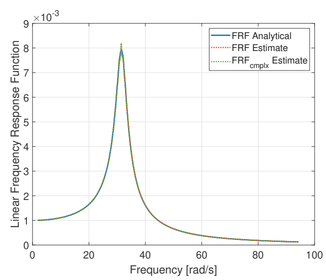
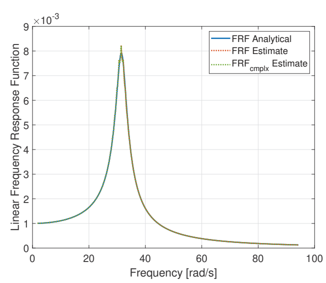
Tab. 53: SCENARIO B104: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B104	101	5.000	0.100	-	-	-	-	0.3	15

Signal	Time domain	Freq. domain
$f(t)$		
$x(t)$		
$v(t)$		

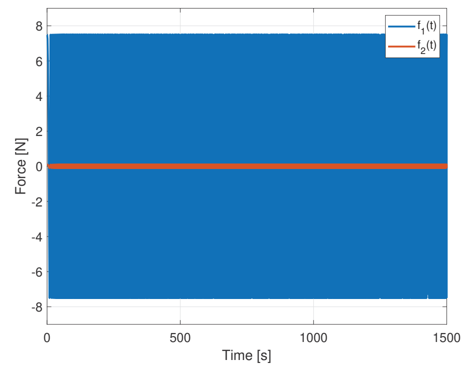
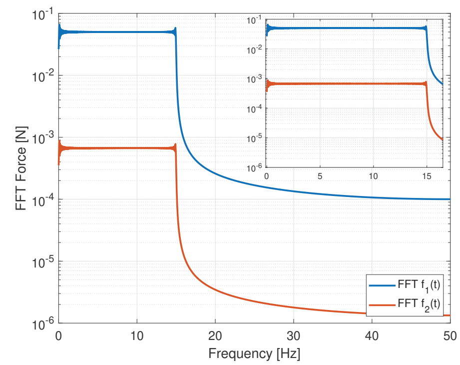
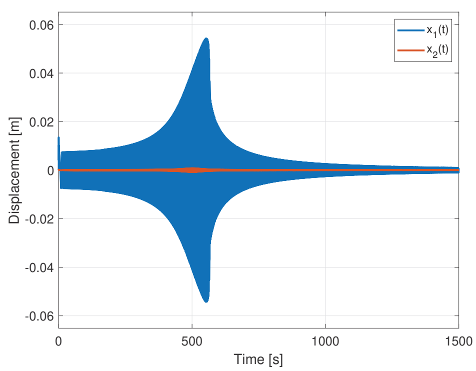
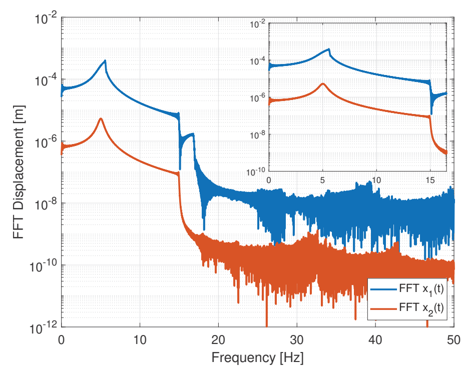
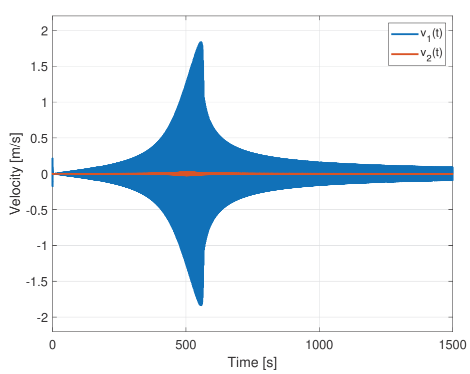
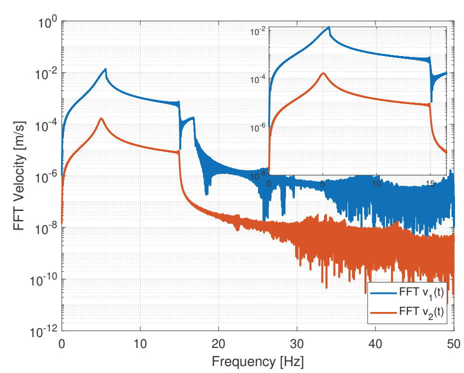
Tab. 54: SCENARIO B104: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B104	101	5.000	0.100	-	-	-	-	0.3	15

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 9.917e+04$ (0.83%) $k_{3,cmplx} = 9.917e{+}04 - i2.893e{+}02$
H$_2$-NIXO		$k_3 = 1.040e+05$ (4.01%) $k_{3,cmplx} = 1.040e{+}05 - i5.453e{+}02$
H$_1$-NIXO with lin. data provided	×	$k_3 = 9.991e+04$ (0.09%) $k_{3,cmplx} = 9.991e{+}04 + i5.840e{+}02$
H$_2$-NIXO with lin. data provided	×	$k_3 = 9.981e+04$ (0.19%) $k_{3,cmplx} = 9.981e{+}04 - i4.633e{+}01$

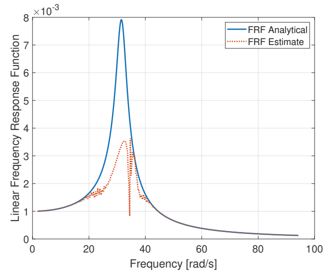
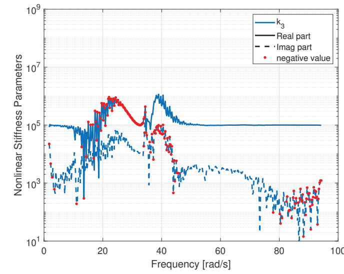
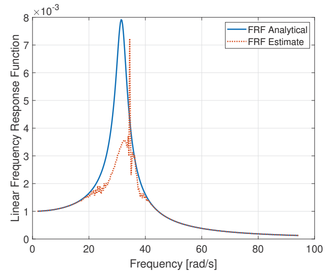
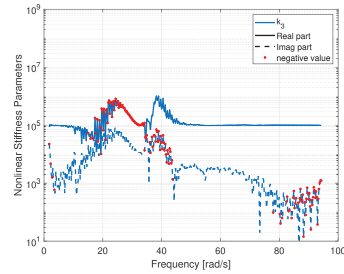
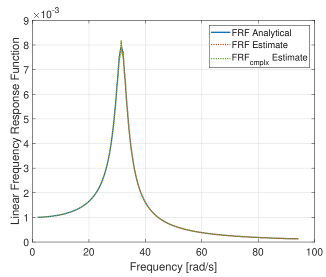
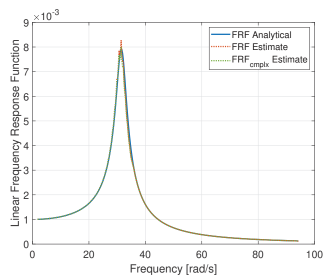
Tab. 55: SCENARIO B105: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B105	101	7.500	0.100	-	-	-	-	0.3	15

Signal	Time domain	Freq. domain
$f(t)$		
$x(t)$		
$v(t)$		

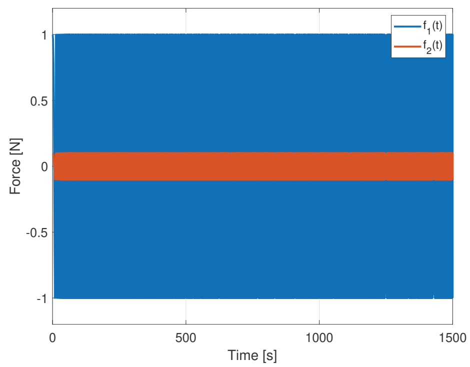
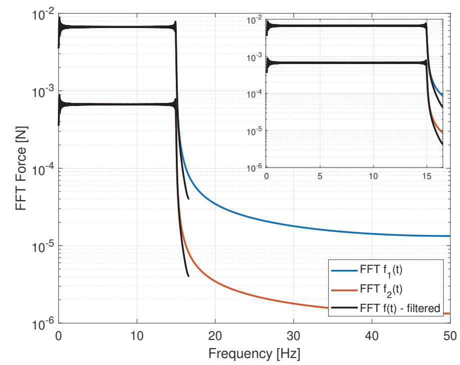
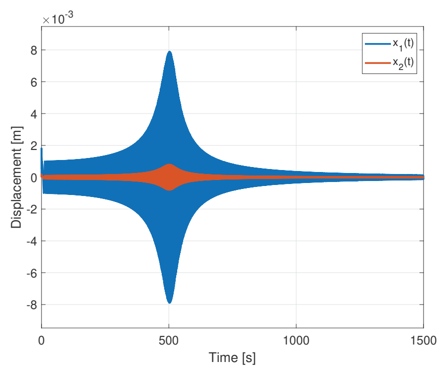
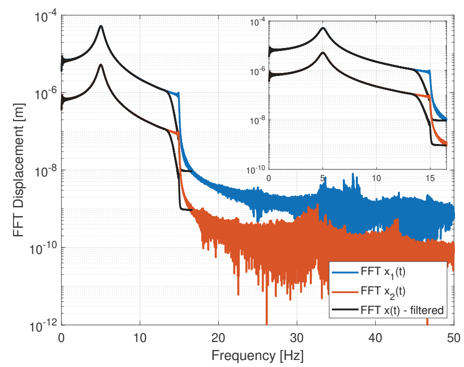
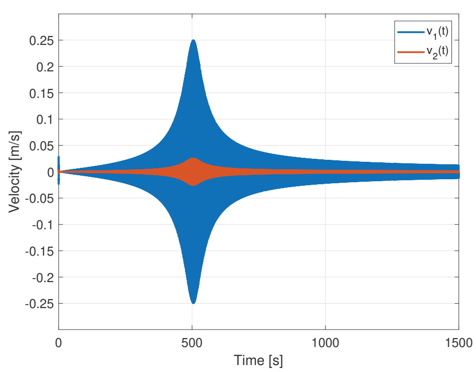
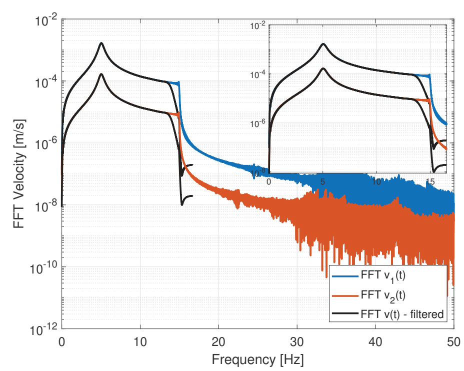
Tab. 56: SCENARIO B105: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B105	101	7.500	0.100	-	-	-	-	0.3	15

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 9.928e+04\ (0.72\%)$ $k_{3,cmplx} = 9.928e{+}04 - i3.033e{+}02$
H$_2$-NIXO		$k_3 = 1.104e+05\ (10.44\%)$ $k_{3,cmplx} = 1.104e{+}05 - i3.270e{+}03$
H$_1$-NIXO with lin. data provided	×	$k_3 = 9.972e+04\ (0.28\%)$ $k_{3,cmplx} = 9.972e{+}04 + i1.491e{+}02$
H$_2$-NIXO with lin. data provided	×	$k_3 = 9.963e+04\ (0.37\%)$ $k_{3,cmplx} = 9.963e{+}04 - i1.388e{+}02$

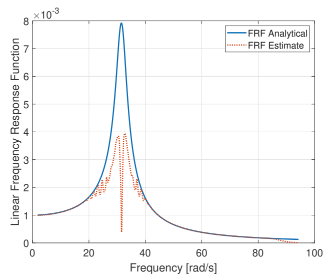
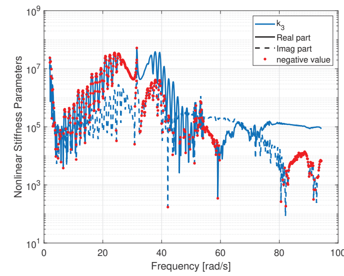
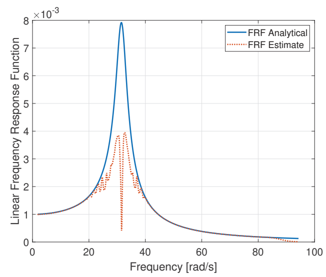
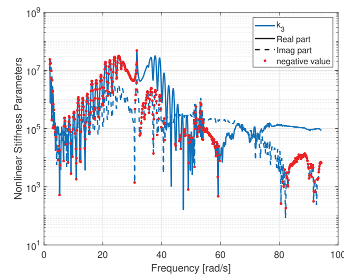
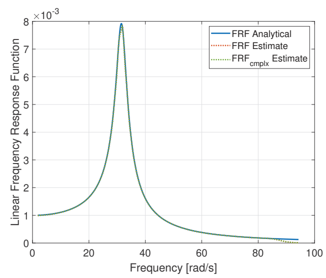
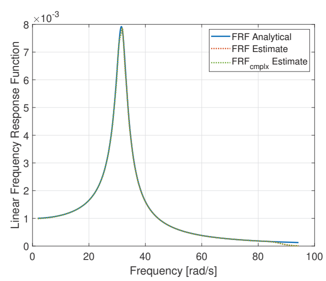
Tab. 57: SCENARIO B106: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B106	102	1.000	0.100	3	-	-	-	0.3	15

Signal	Time domain	Freq. domain
$f(t)$		
$x(t)$		
$v(t)$		

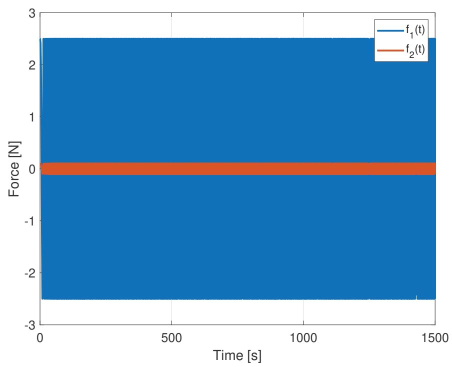
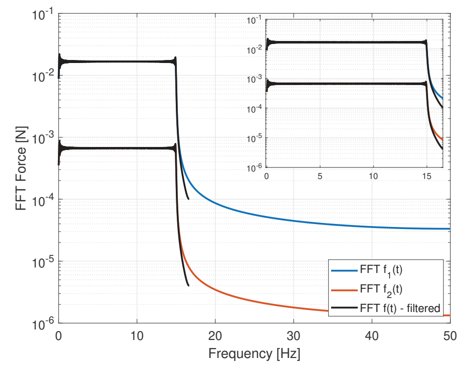
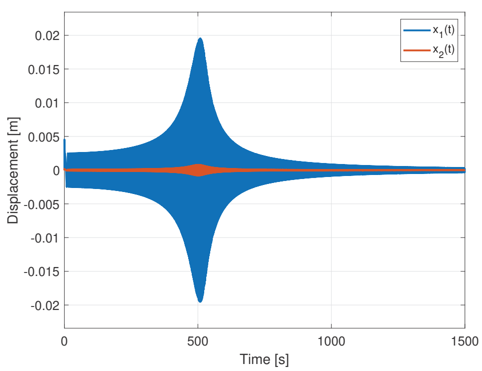
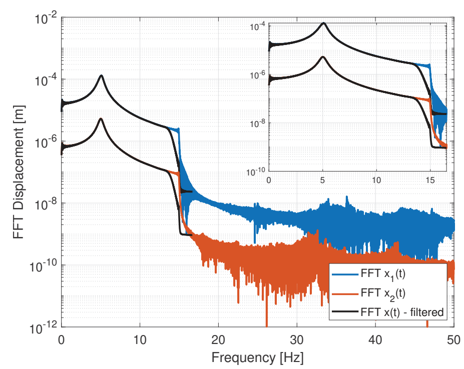
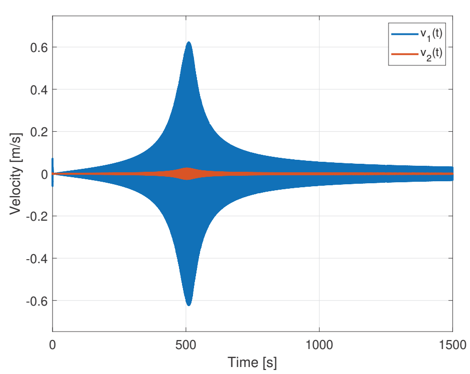
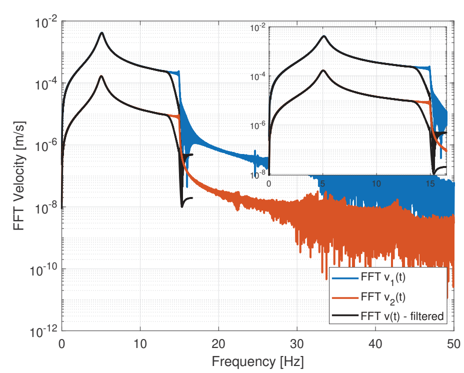
Tab. 58: SCENARIO B106: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B106	102	1.000	0.100	3	-	-	-	0.3	15

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
$\mathbf{H}_1$-NIFO		
$\mathbf{H}_2$-NIFO		
$\mathbf{H}_1$-NIXO		$k_3 = 9.911e+04$ (0.89%) $k_{3,cmplx} = 9.911e{+}04 - i5.672e{+}02$
$\mathbf{H}_2$-NIXO		$k_3 = 9.691e+04$ (3.09%) $k_{3,cmplx} = 9.692e{+}04 - i2.667e{+}03$
$\mathbf{H}_1$-NIXO with lin. data provided	×	$k_3 = 1.096e+05$ (9.62%) $k_{3,cmplx} = 1.096e{+}05 + i4.533e{+}04$
$\mathbf{H}_2$-NIXO with lin. data provided	×	$k_3 = 1.090e+05$ (9.00%) $k_{3,cmplx} = 1.090e{+}05 + i3.832e{+}04$

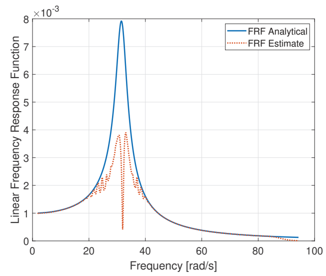
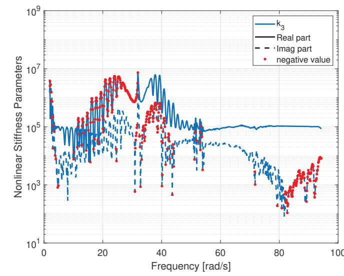
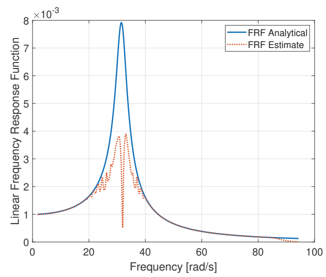
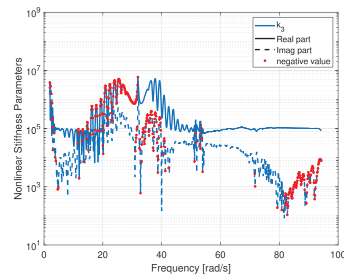
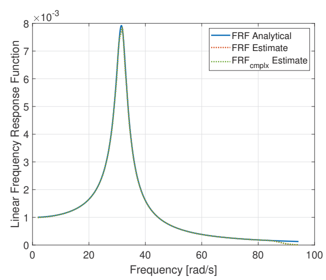
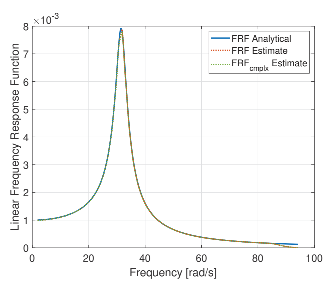
Tab. 59: SCENARIO B107: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B107	102	2.500	0.100	3	-	-	-	0.3	15

Signal	Time domain	Freq. domain
$f(t)$		
$x(t)$		
$v(t)$		

Tab. 60: SCENARIO B107: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B107	102	2.500	0.100	3	-	-	-	0.3	15

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 1.023e+05$ (2.32%) $k_{3,cmplx} = 1.023e{+}05 - i2.661e{+}02$
H$_2$-NIXO		$k_3 = 1.009e+05$ (0.94%) $k_{3,cmplx} = 1.009e{+}05 - i3.907e{+}03$
H$_1$-NIXO with lin. data provided	×	$k_3 = 1.035e+05$ (3.50%) $k_{3,cmplx} = 1.035e{+}05 + i7.284e{+}03$
H$_2$-NIXO with lin. data provided	×	$k_3 = 1.033e+05$ (3.32%) $k_{3,cmplx} = 1.033e{+}05 + i6.087e{+}03$

Comments

1. Comments to this section are very similar to those on page 25.

2. One additional comment should be added nevertheless. The NIFO methods fail to estimate the linear frequency response function. However, they succeed in finding the nonlinear parameter values for the off-resonant frequency range. NIXO methods return accurate estimates of both: linear FRF and nonlinear parameter k_3.

2.2.2 Forcing Type: Burst Random

Tab. 61: Parameters describing broad band burst random forcing function and Butterworth filter used to process the input signal. Example 2.

F [N]	t_{st} [s]	t_{end} [s]	Δt [s]	Burst Start	Burst End
different values	0	512.5	0.01	0%	100%

Ω_1^{filt} [Hz]	Ω_2^{filt} [Hz]	Decimation Factor	Filter Order
different values		**different values**	

Tab. 62: Description of various scenarios the algorithms are tested with. Example 2.

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}	I/O Signals	Results
B201	111	1.0	0.1	2	8	0.3	15	0.3	15	Tab. 63	Tab. 64
B202		2.5								Tab. 65	Tab. 66
B203		5.0								Tab. 67	Tab. 68
B204		7.5								Tab. 69	Tab. 70
B205		10.0								Tab. 71	Tab. 72
B206		15.0								Tab. 73	Tab. 74
B207		20.0								Tab. 75	Tab. 76
B208		25.0								Tab. 77	Tab. 78
B209		30.0								Tab. 79	Tab. 80
B210	112	5	0.1	3	8	0.3	15	0.3	15	Tab. 81	Tab. 82
B211		7.5								Tab. 83	Tab. 84
B212		10.0								Tab. 85	Tab. 86
B213		30.0								Tab. 87	Tab. 88
B214	113	5	0.1	-	-	-	-	0.3	15	Tab. 89	Tab. 90
B215		7.5								Tab. 91	Tab. 92
B216		10.0								Tab. 93	Tab. 94
B217		30.0								Tab. 95	Tab. 96

Force amplitudes expressed in newtons, frequencies in hertz.

DF – Decimation Factor

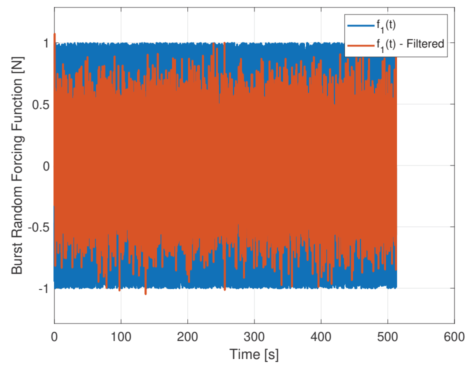
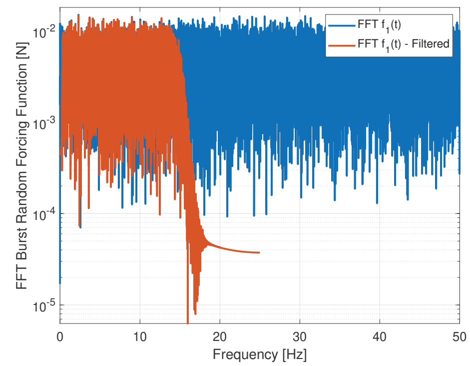
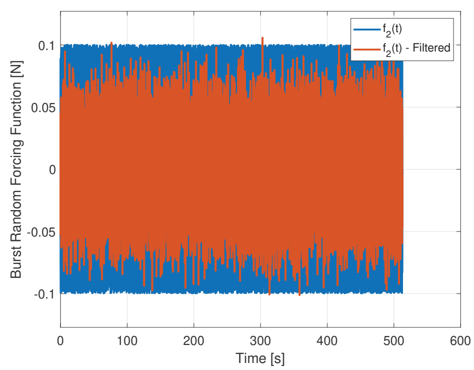
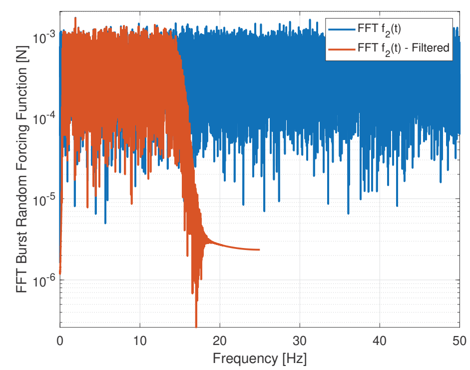
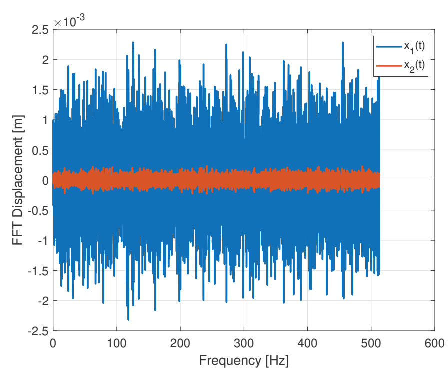
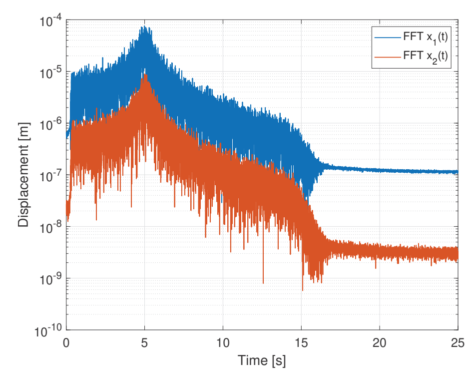
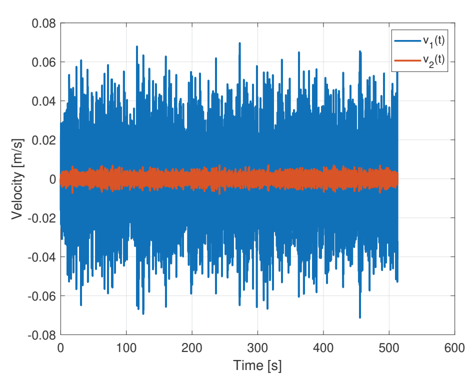
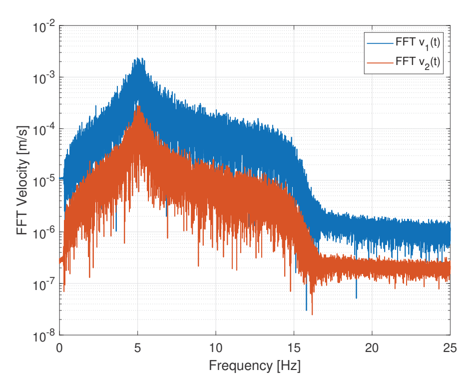
FO – Butterworth Filter Order

Ω^{filt} – cut-off frequency; Ω_1^{filt} and Ω_2^{filt} are lower and upper cut-off frequencies, respectively

Ω^{spect} – auto- and cross-spectra are computed for frequency range (Ω_1^{spect}, Ω_2^{spect}).

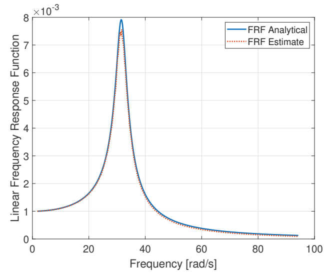
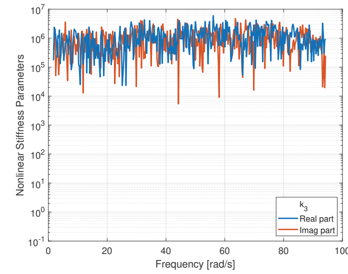
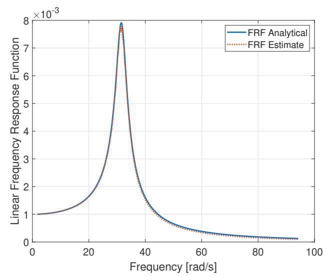
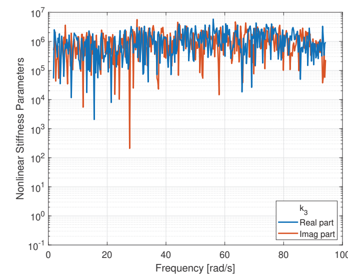
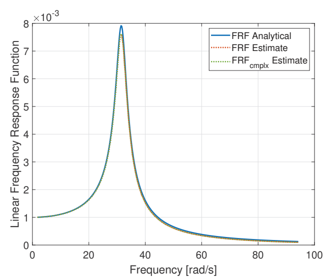
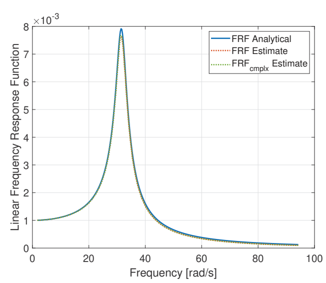
Tab. 63: SCENARIO B201: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B201	111	1.000	0.100	2	8	0.3	15	0.3	15

Signal	Time domain	Freq. domain
$f_I(t)$		
$f_{II}(t)$		
$x(t)$		
$v(t)$		

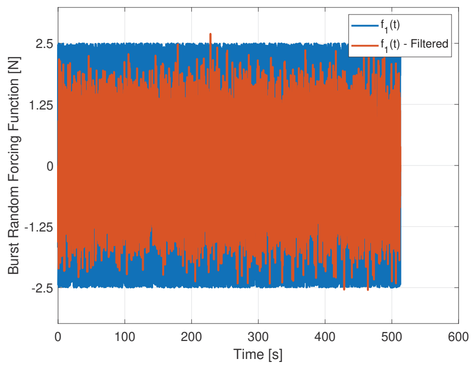
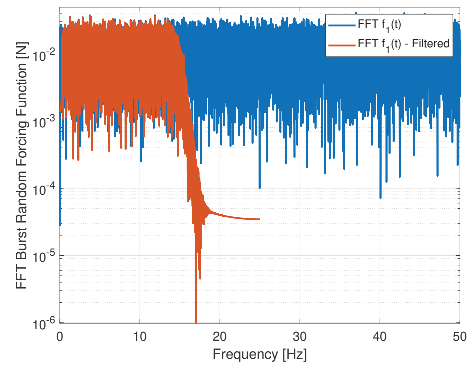
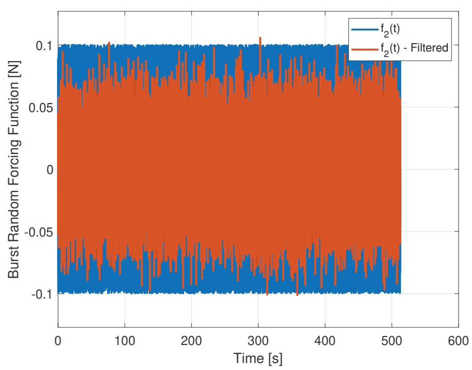
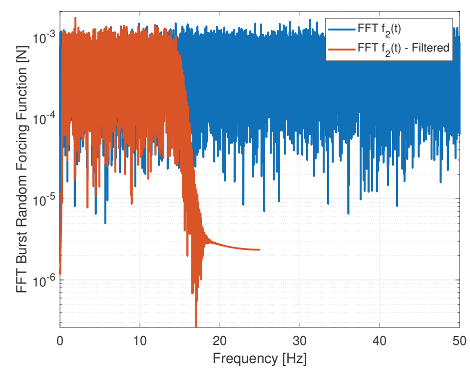
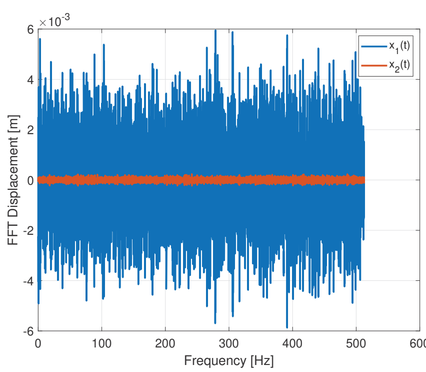
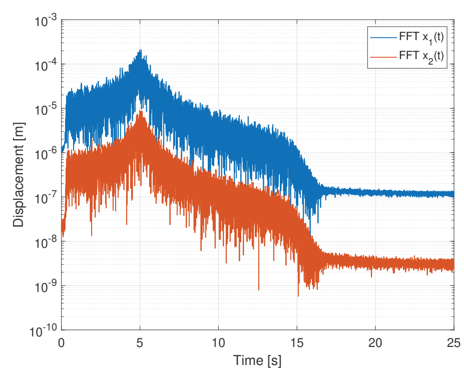
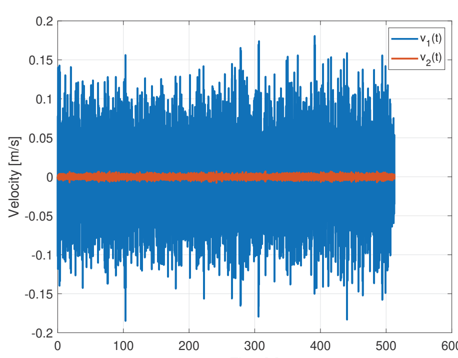
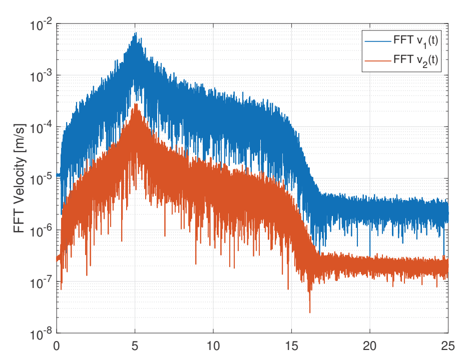
Tab. 64: SCENARIO B201: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B201	111	1.000	0.100	2	8	0.3	15	0.3	15

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 1.689e+05$ (68.88%) $k_{3,cmplx} = 1.689e+05 \; + \; i2.772e+05$
H$_2$-NIXO		$k_3 = 6.167e+04$ (38.33%) $k_{3,cmplx} = 5.248e+04 \; + \; i8.165e+05$
H$_1$-NIXO with lin. data provided	×	$k_3 = -3.862e+06$ (3961.94%) $k_{3,cmplx} = -3.862e+06 + i4.299e+06$
H$_2$-NIXO with lin. data provided	×	$k_3 = 5.377e+05$ (437.69%) $k_{3,cmplx} = 5.377e+05 \; + \; i3.925e+06$

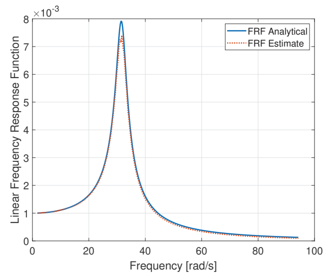
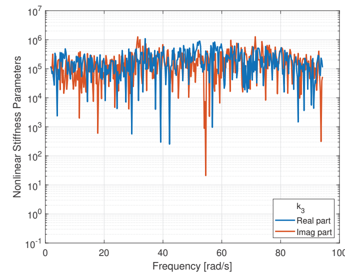
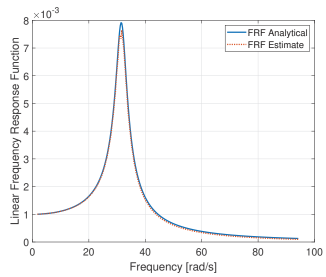
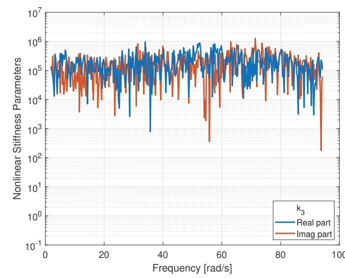
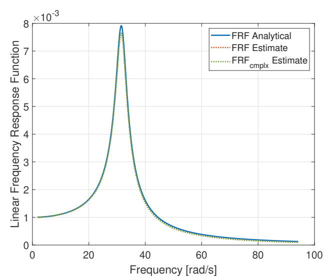
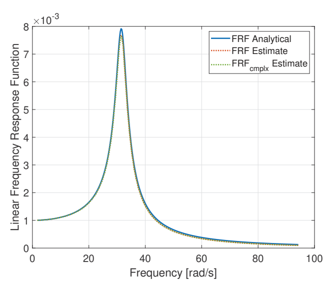
Tab. 65: SCENARIO B202: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B202	111	2.500	0.100	2	8	0.3	15	0.3	15

Signal	Time domain	Freq. domain
$f_I(t)$		
$f_{II}(t)$		
$x(t)$		
$v(t)$		

Tab. 66: SCENARIO B202: Results

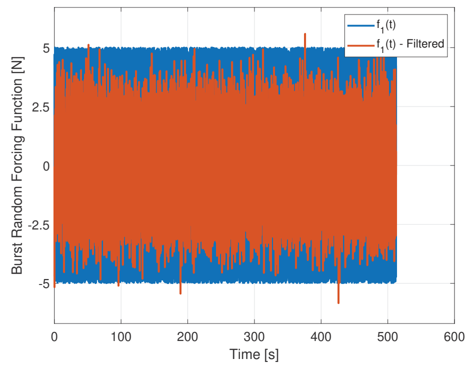
Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B202	111	2.500	0.100	2	8	0.3	15	0.3	15

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 8.384e+04$ (16.16%) $k_{3,cmplx} = 8.384e+04 + i2.045e+02$
H$_2$-NIXO		$k_3 = 9.478e+04$ (5.22%) $k_{3,cmplx} = 9.470e+04 - i7.017e+04$
H$_1$-NIXO with lin. data provided	×	$k_3 = -4.961e+05$ (596.15%) $k_{3,cmplx} = -4.961e+05 + i6.148e+05$
H$_2$-NIXO with lin. data provided	×	$k_3 = 1.810e+05$ (81.01%) $k_{3,cmplx} = 1.810e+05 + i5.718e+05$

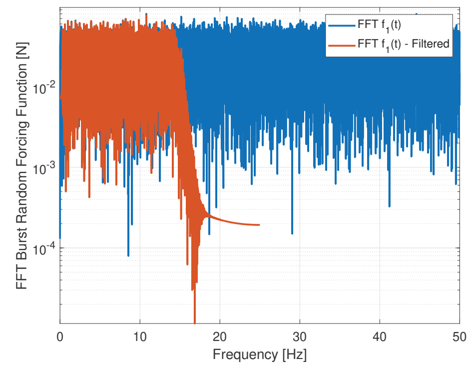
Tab. 67: SCENARIO B203: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B203	111	5.000	0.100	2	8	0.3	15	0.3	15

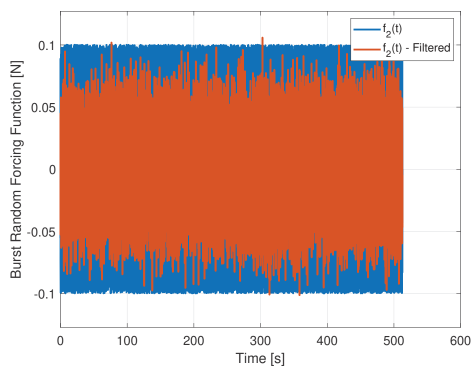
<table>
<tr><th>Signal</th><th>Time domain</th><th>Freq. domain</th></tr>
<tr><td>$f_I(t)$</td><td>



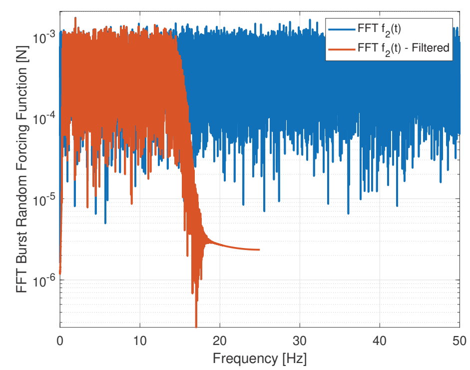
</td><td>



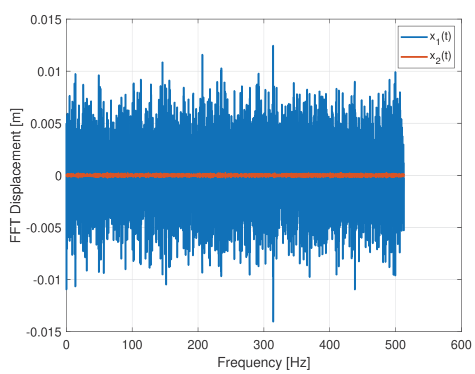
</td></tr>
<tr><td>$f_{II}(t)$</td><td>



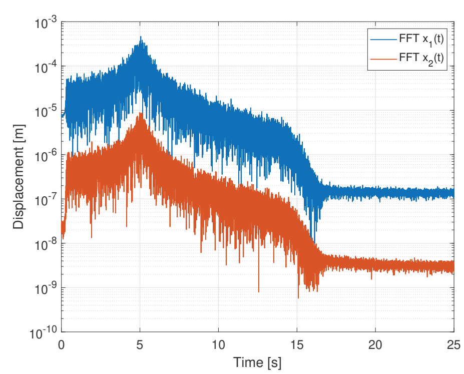
</td><td>



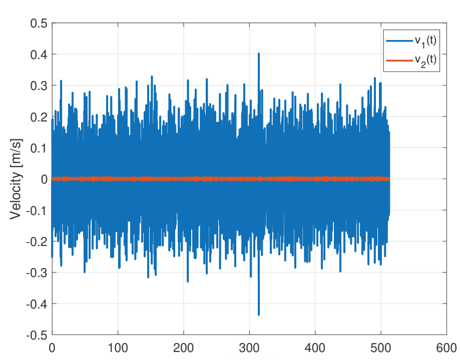
</td></tr>
<tr><td>$x(t)$</td><td>



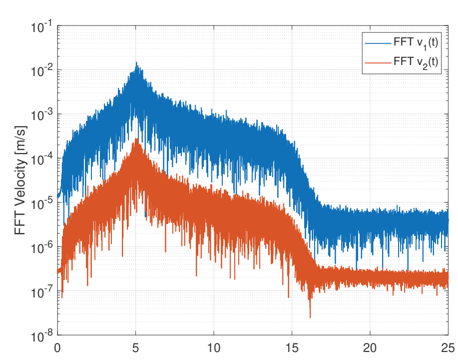
</td><td>



</td></tr>
<tr><td>$v(t)$</td><td>



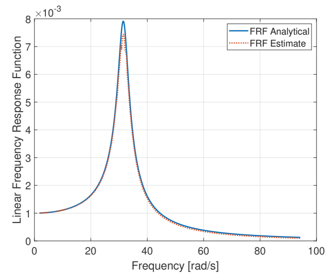
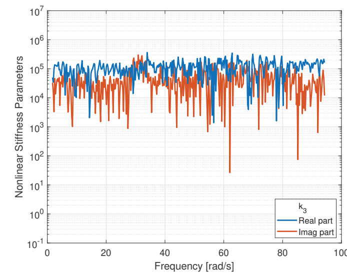
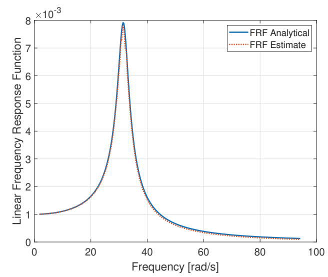
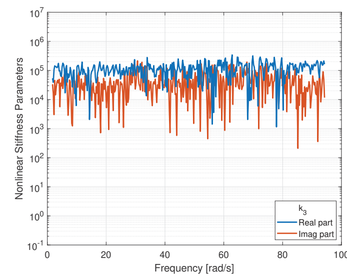
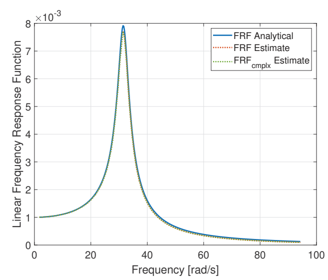
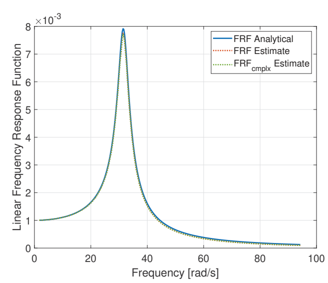
</td><td>



</td></tr>
</table>

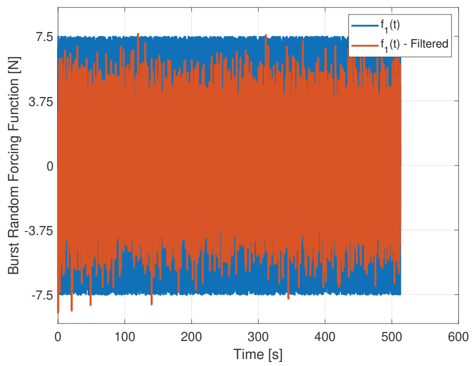
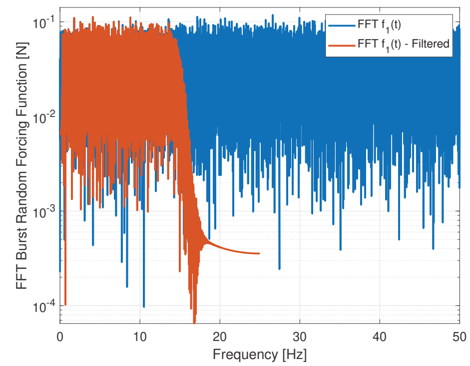
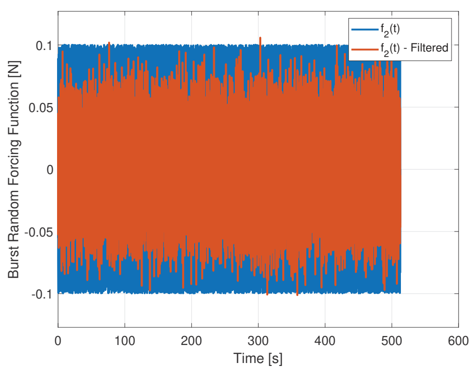
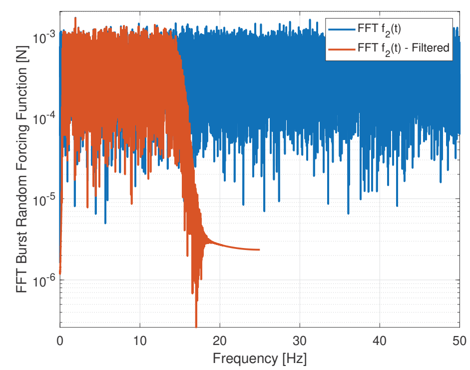
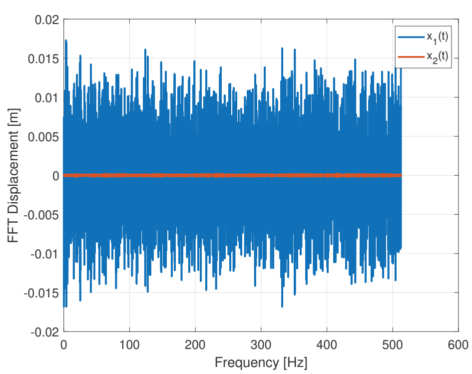
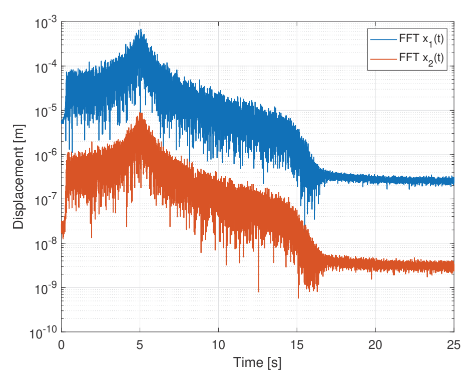
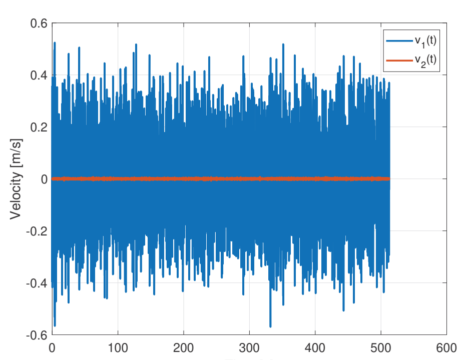
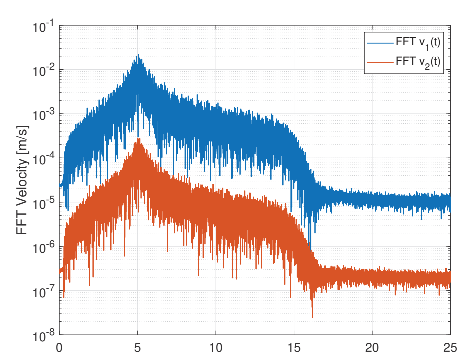
Tab. 68: SCENARIO B203: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B203	111	5.000	0.100	2	8	0.3	15	0.3	15

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
$\mathbf{H}_1$-NIFO		
$\mathbf{H}_2$-NIFO		
$\mathbf{H}_1$-NIXO		$k_3 = 9.529e+04$ (4.71%) $k_{3,cmplx} = 9.529e+04 - i6.043e+03$
$\mathbf{H}_2$-NIXO		$k_3 = 1.071e+05$ (7.13%) $k_{3,cmplx} = 1.071e+05 - i2.267e+04$
$\mathbf{H}_1$-NIXO with lin. data provided	×	$k_3 = -1.053e+05$ (205.31%) $k_{3,cmplx} = -1.053e+05 + i1.548e+05$
$\mathbf{H}_2$-NIXO with lin. data provided	×	$k_3 = 1.154e+05$ (15.40%) $k_{3,cmplx} = 1.154e+05 + i1.515e+05$

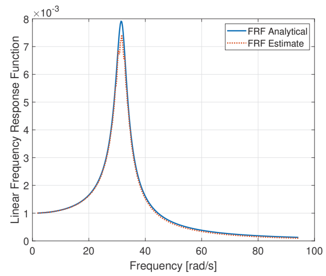
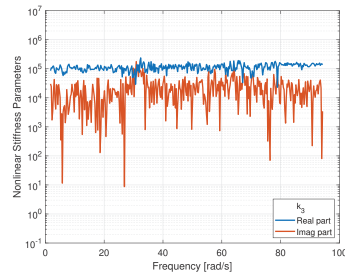
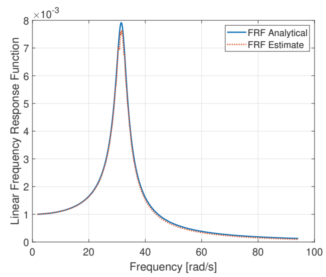
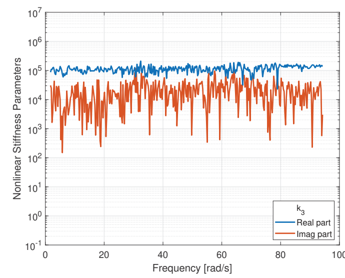
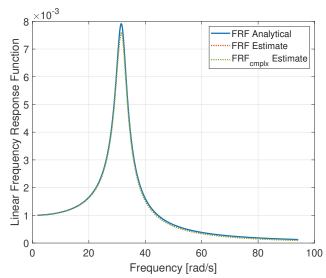
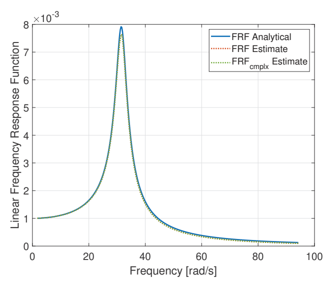
Tab. 69: SCENARIO B204: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B204	111	7.500	0.100	2	8	0.3	15	0.3	15

Signal	Time domain	Freq. domain
$f_I(t)$		
$f_{II}(t)$		
$x(t)$		
$v(t)$		

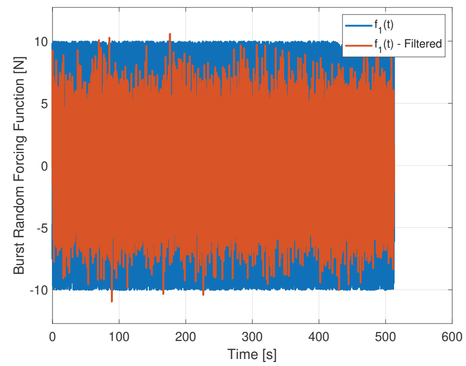
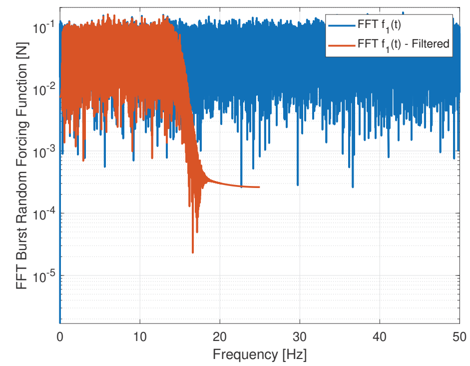
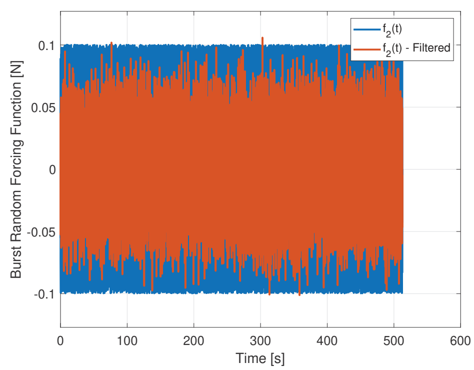
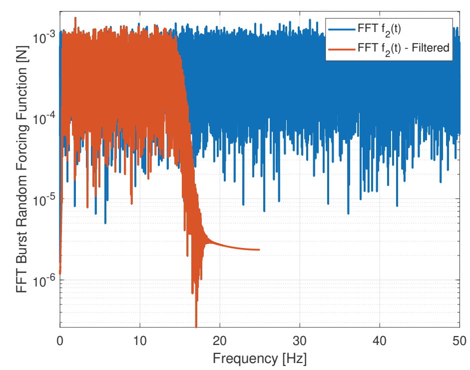
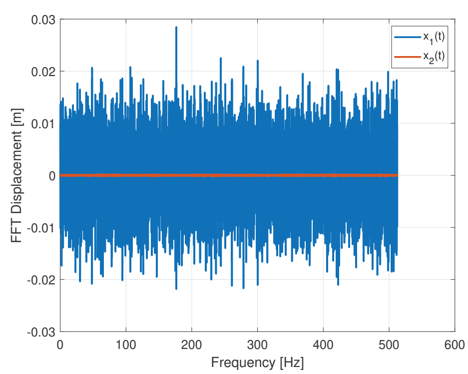
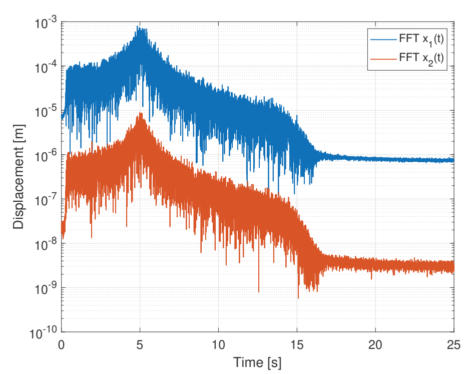
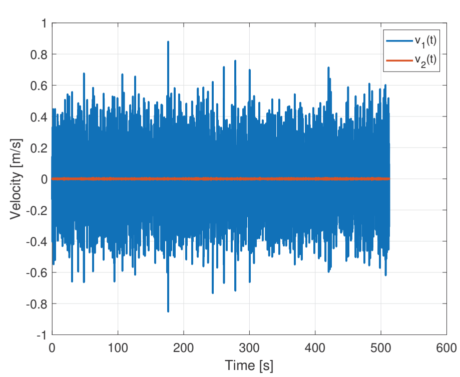
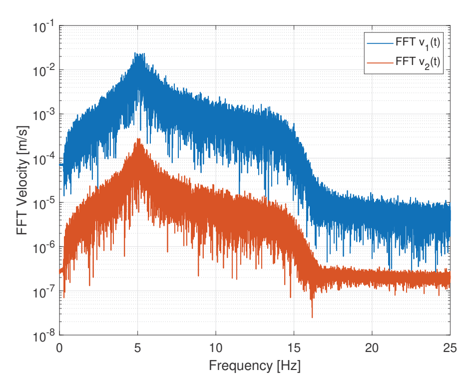
Tab. 70: SCENARIO B204: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B204	111	7.500	0.100	2	8	0.3	15	0.3	15

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 9.918e+04$ (0.82%) $k_{3,cmplx} = 9.918e+04 \; + \; i4.922e+02$
H$_2$-NIXO		$k_3 = 1.060e+05$ (6.01%) $k_{3,cmplx} = 1.060e+05 \; - \; i7.059e+02$
H$_1$-NIXO with lin. data provided	×	$k_3 = 3.870e+04$ (61.30%) $k_{3,cmplx} = 3.870e+04 \; + \; i7.047e+04$
H$_2$-NIXO with lin. data provided	×	$k_3 = 1.048e+05$ (4.77%) $k_{3,cmplx} = 1.048e+05 \; + \; i6.515e+04$

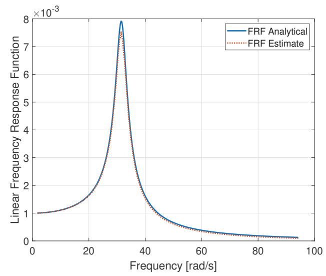
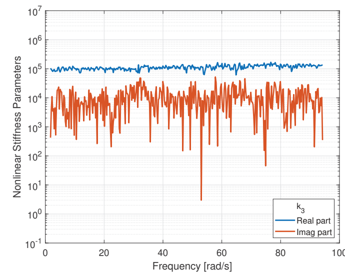
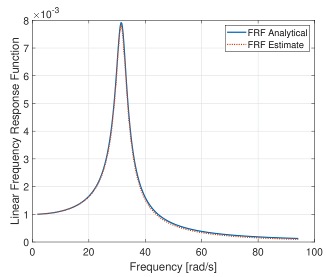
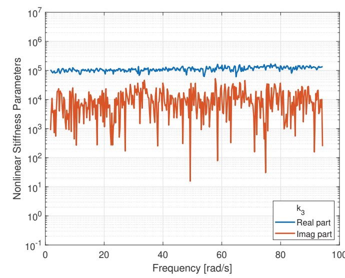
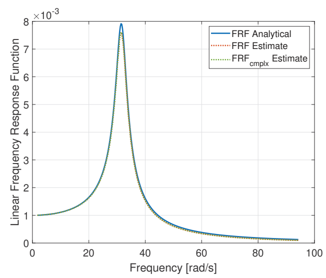
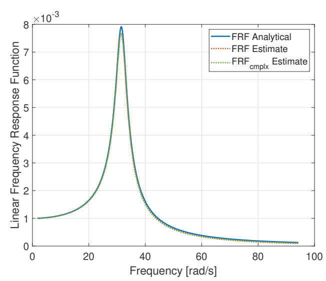
Tab. 71: SCENARIO B205: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B205	111	10.000	0.100	2	8	0.3	15	0.3	15

Signal	Time domain	Freq. domain
$f_I(t)$		
$f_{II}(t)$		
$x(t)$		
$v(t)$		

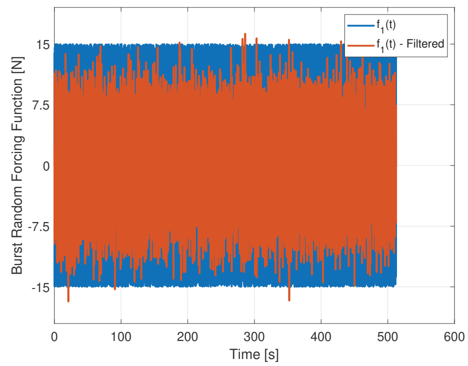
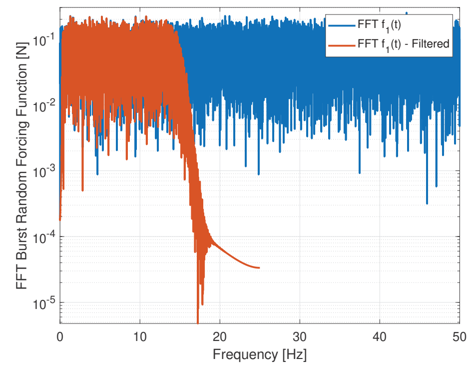
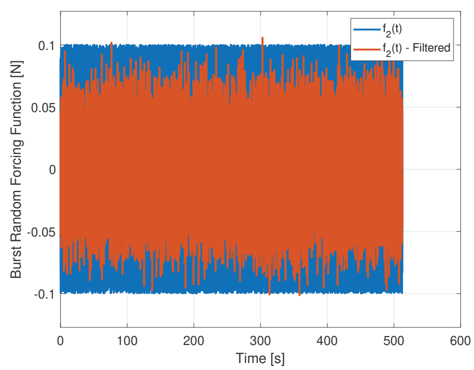
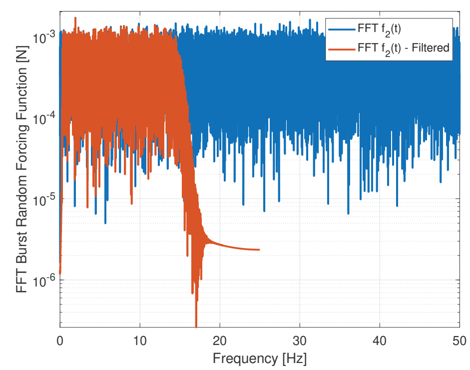
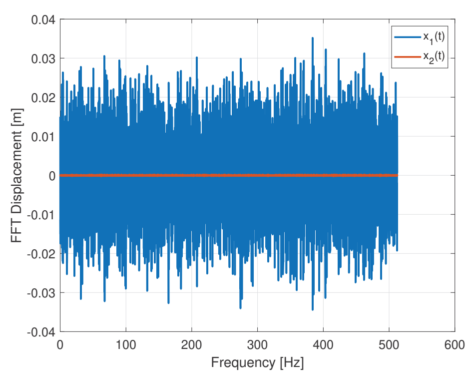
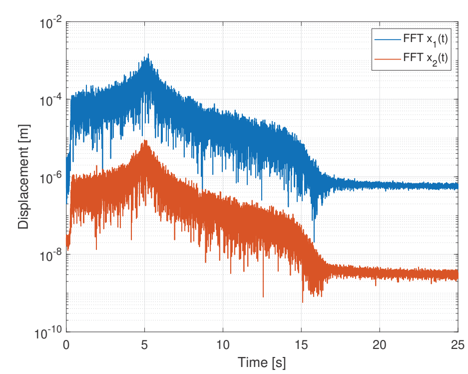
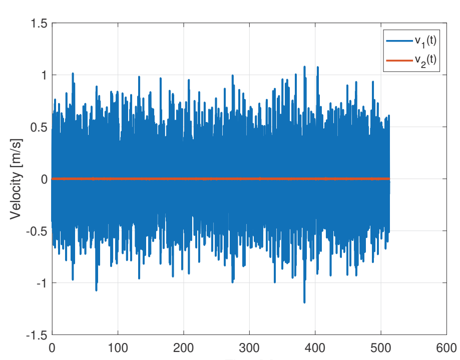
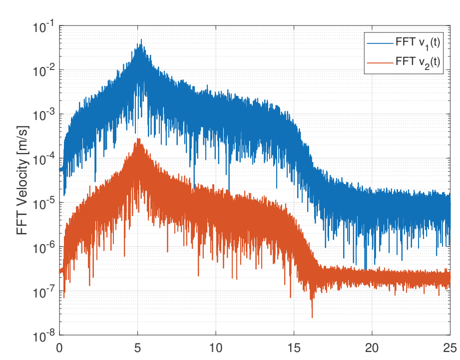
Tab. 72: SCENARIO B205: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B205	111	10.000	0.100	2	8	0.3	15	0.3	15

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 9.914e+04$ (0.86%) $k_{3,cmplx} = 9.914e+04 + i1.813e+03$
H$_2$-NIXO		$k_3 = 1.047e+05$ (4.73%) $k_{3,cmplx} = 1.047e+05 + i5.614e+03$
H$_1$-NIXO with lin. data provided	×	$k_3 = 4.681e+04$ (53.19%) $k_{3,cmplx} = 4.681e+04 + i4.682e+04$
H$_2$-NIXO with lin. data provided	×	$k_3 = 9.880e+04$ (1.20%) $k_{3,cmplx} = 9.880e+04 + i4.117e+04$

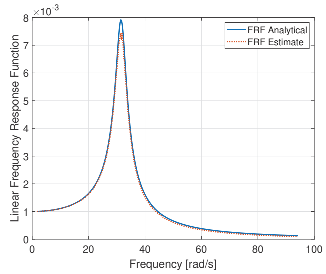
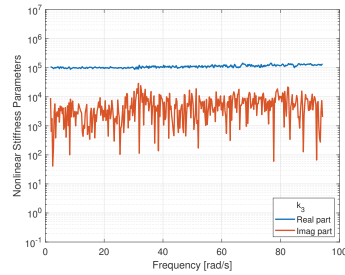
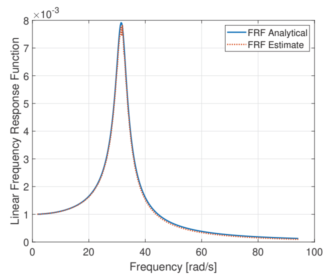
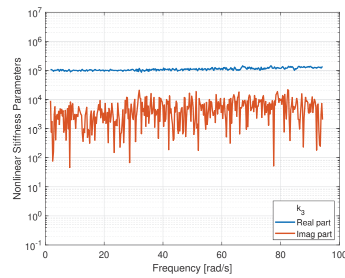
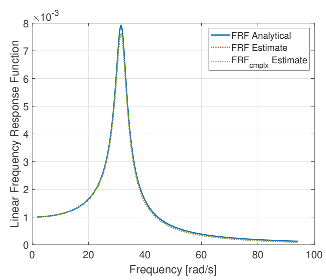
Tab. 73: SCENARIO B206: Input/Output Signals

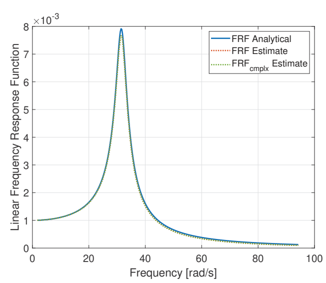
Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B206	111	15.000	0.100	2	8	0.3	15	0.3	15

<table>
<tr><th>Signal</th><th>Time domain</th><th>Freq. domain</th></tr>
<tr><td>$f_I(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$f_{II}(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$x(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$v(t)$</td><td>
</td><td>
</td></tr>
</table>

Tab. 74: SCENARIO B206: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B206	111	15.000	0.100	2	8	0.3	15	0.3	15

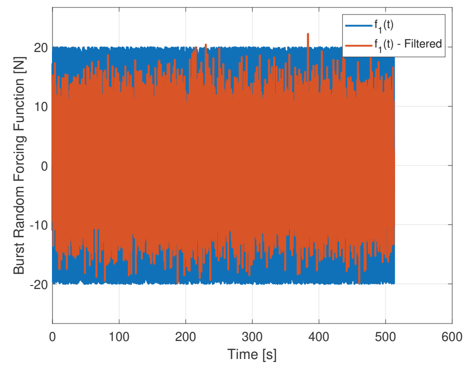
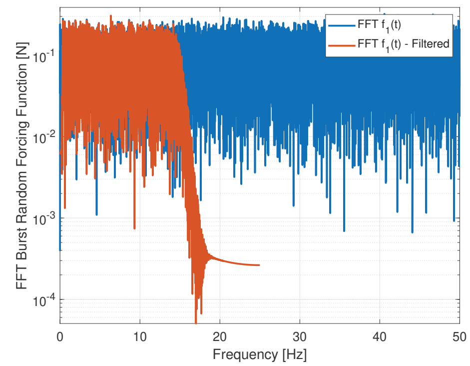
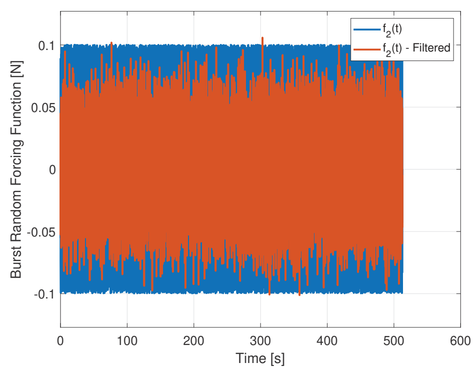
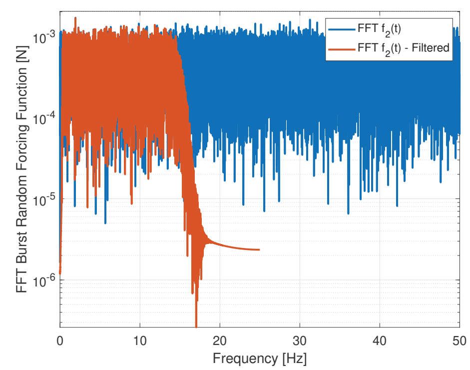
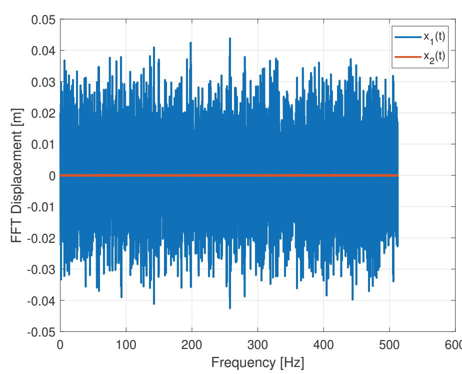
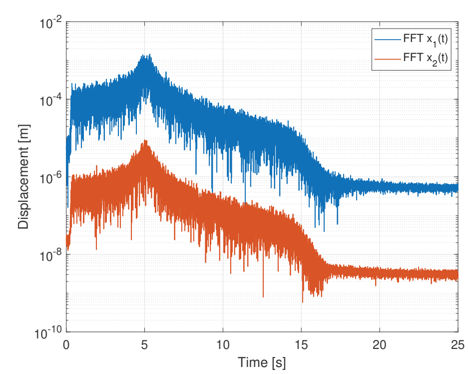
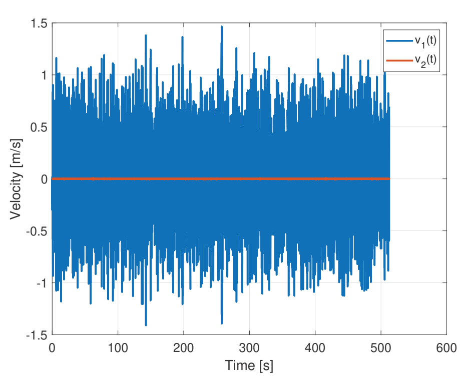
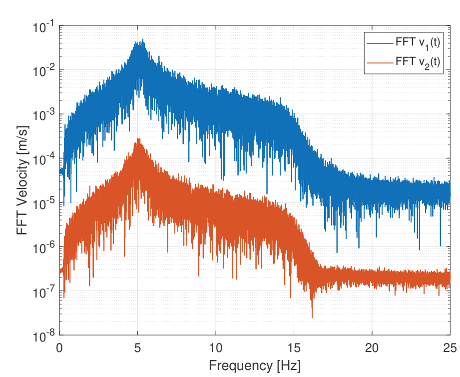
<table>
<tr><th>Method</th><th>Linear FRF estimate</th><th>Nonlin. param. estimates
(rel. error)</th></tr>
<tr><td>H$_1$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_2$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_1$-NIXO</td><td>
</td><td>$k_3 = 1.014e+05$ (1.38%)

$k_{3,cmplx} = 1.014e+05 + i8.871e+01$</td></tr>
<tr><td>H$_2$-NIXO</td><td>
</td><td>$k_3 = 1.031e+05$ (3.15%)

$k_{3,cmplx} = 1.031e+05 + i2.242e+02$</td></tr>
<tr><td>H$_1$-NIXO
with lin. data
provided</td><td>×</td><td>$k_3 = 7.885e+04$ (21.15%)
$k_{3,cmplx} = 7.885e+04 + i1.815e+04$</td></tr>
<tr><td>H$_2$-NIXO
with lin. data
provided</td><td>×</td><td>$k_3 = 9.877e+04$ (1.23%)
$k_{3,cmplx} = 9.877e+04 + i1.687e+04$</td></tr>
</table>

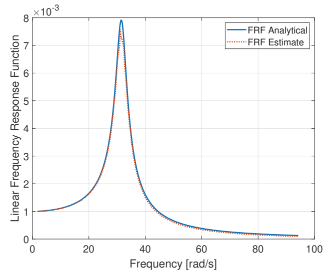
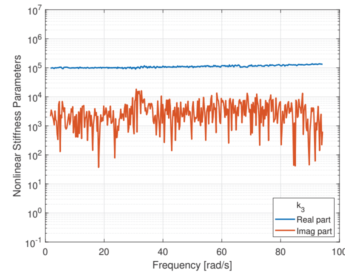
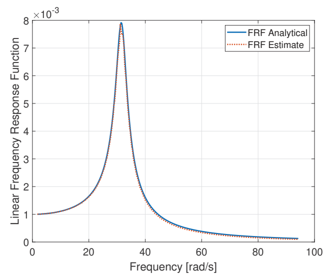
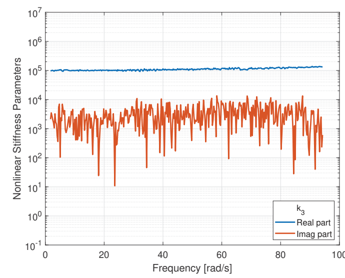
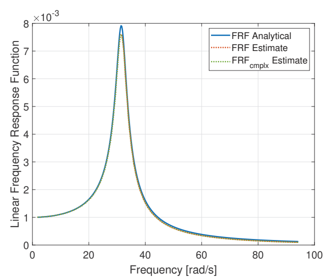
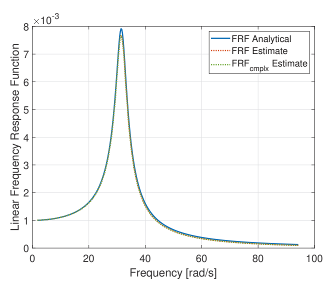
Tab. 75: SCENARIO B207: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B207	111	20.000	0.100	2	8	0.3	15	0.3	15

Signal	Time domain	Freq. domain
$f_I(t)$		
$f_{II}(t)$		
$x(t)$		
$v(t)$		

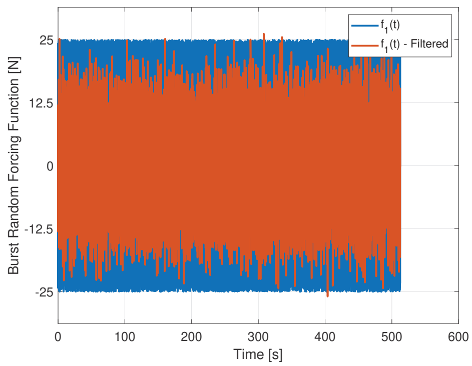
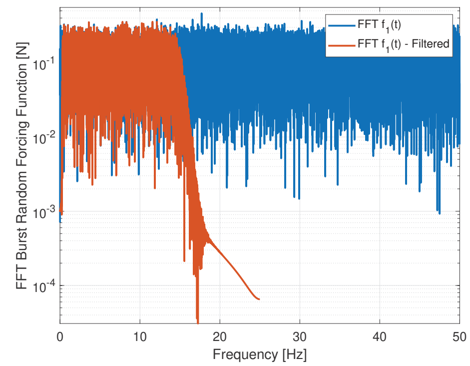
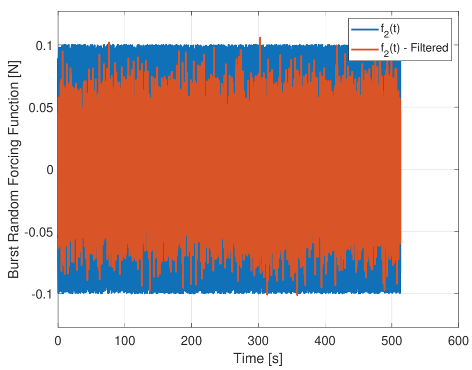
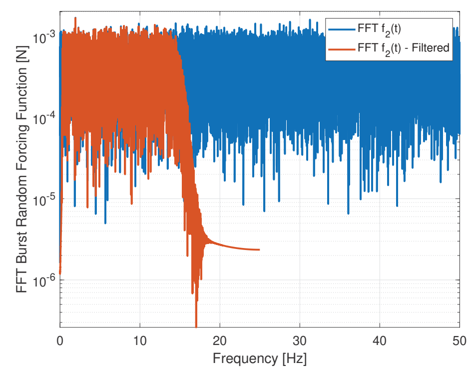
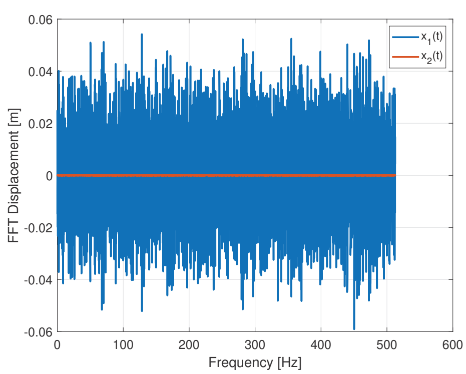
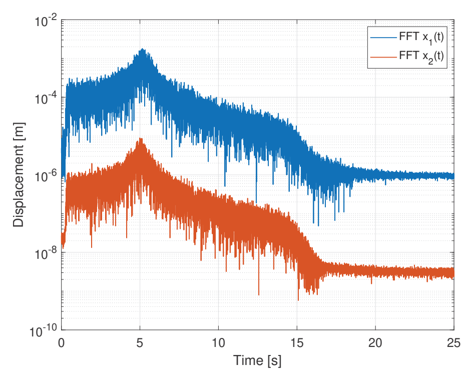
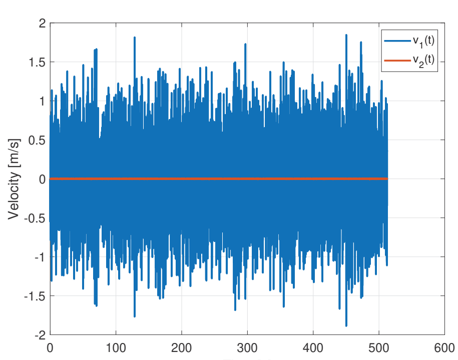
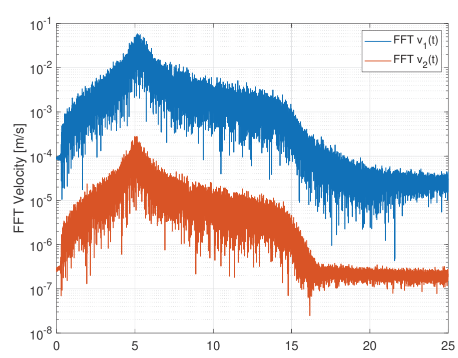
Tab. 76: SCENARIO B207: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B207	111	20.000	0.100	2	8	0.3	15	0.3	15

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 1.021e+05$ (2.10%) $k_{3,cmplx} = 1.021e+05 - i6.292e+02$
H$_2$-NIXO		$k_3 = 1.042e+05$ (4.20%) $k_{3,cmplx} = 1.042e+05 - i4.217e+02$
H$_1$-NIXO with lin. data provided	×	$k_3 = 8.671e+04$ (13.29%) $k_{3,cmplx} = 8.671e+04 + i1.033e+04$
H$_2$-NIXO with lin. data provided	×	$k_3 = 9.983e+04$ (0.17%) $k_{3,cmplx} = 9.983e+04 + i9.221e+03$

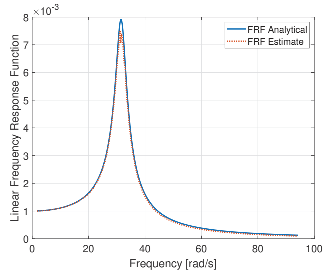
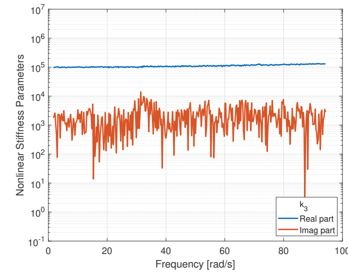
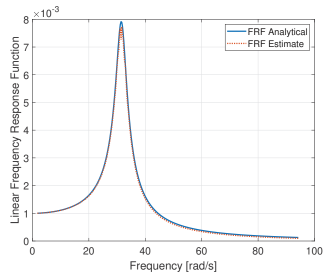
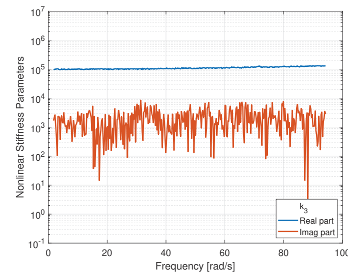
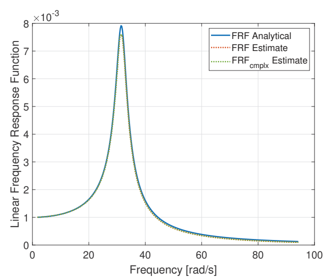
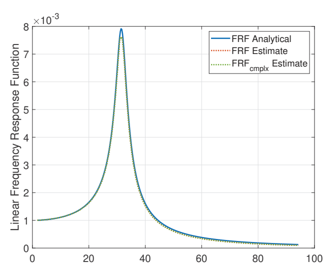
Tab. 77: SCENARIO B208: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B208	111	25.000	0.100	2	8	0.3	15	0.3	15

Signal	Time domain	Freq. domain
$f_I(t)$		
$f_{II}(t)$		
$x(t)$		
$v(t)$		

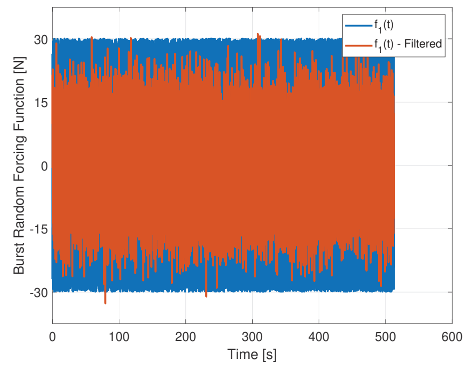
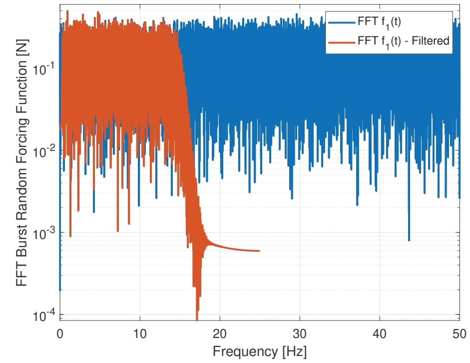
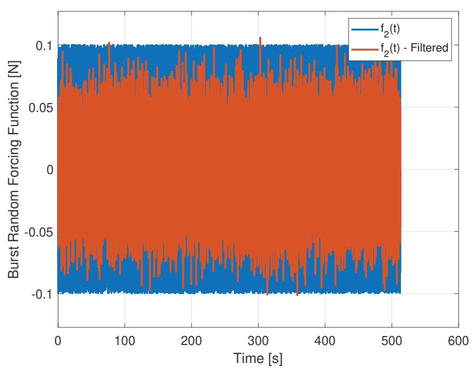
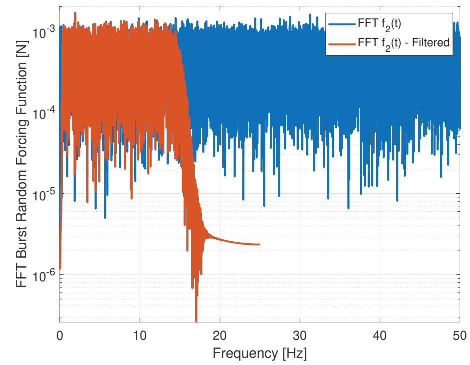
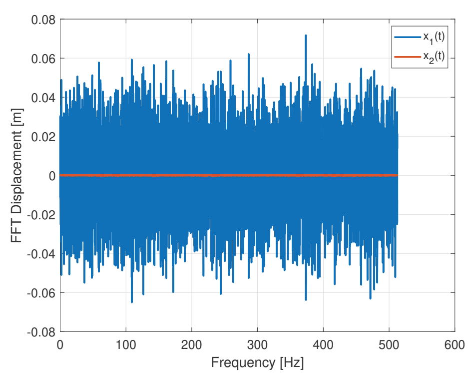
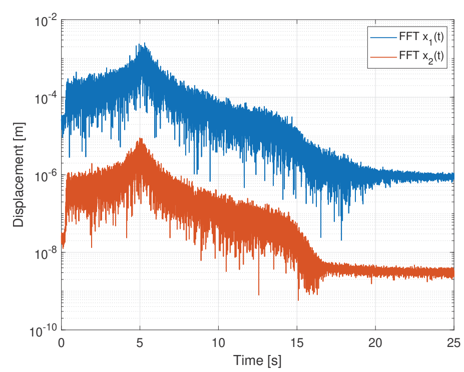
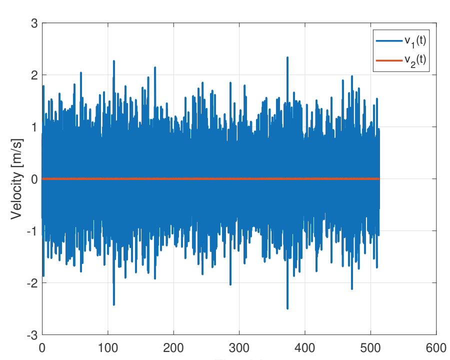
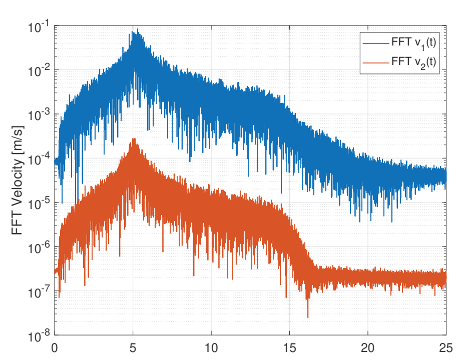
Tab. 78: SCENARIO B208: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B208	111	25.000	0.100	2	8	0.3	15	0.3	15

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 1.024e+05$ (2.43%) $k_{3,cmplx} = 1.024e+05 + i2.025e+02$
H$_2$-NIXO		$k_3 = 1.040e+05$ (3.95%) $k_{3,cmplx} = 1.040e+05 - i1.066e+02$
H$_1$-NIXO with lin. data provided	×	$k_3 = 9.263e+04$ (7.37%) $k_{3,cmplx} = 9.263e+04 + i7.777e+03$
H$_2$-NIXO with lin. data provided	×	$k_3 = 1.003e+05$ (0.25%) $k_{3,cmplx} = 1.003e+05 + i6.839e+03$

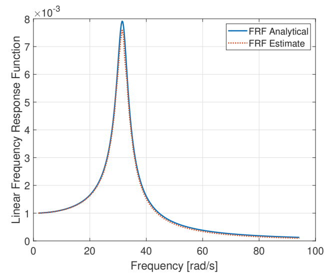
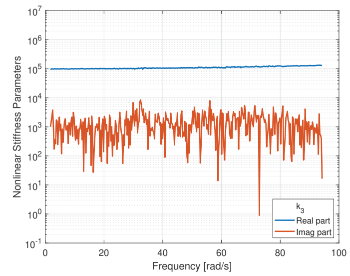
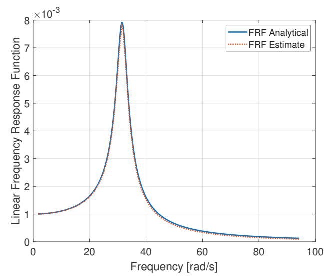
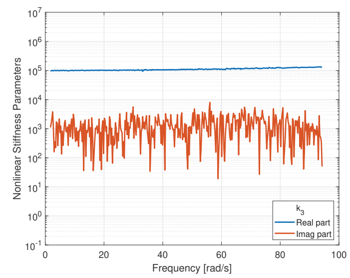
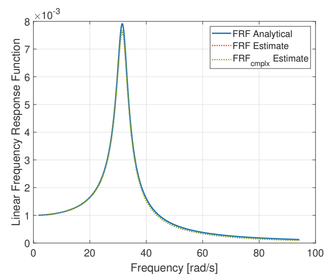
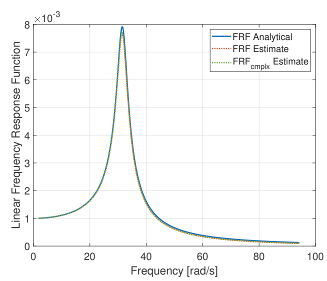
Tab. 79: SCENARIO B209: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B209	111	30.000	0.100	2	8	0.3	15	0.3	15

Signal	Time domain	Freq. domain
$f_I(t)$		
$f_{II}(t)$		
$x(t)$		
$v(t)$		

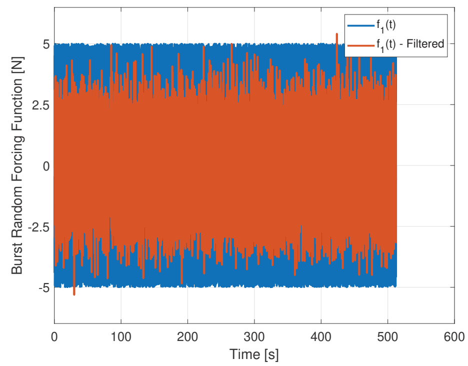
Tab. 80: SCENARIO B209: Results

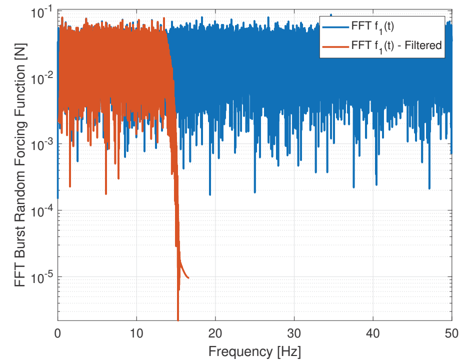
Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B209	111	30.000	0.100	2	8	0.3	15	0.3	15

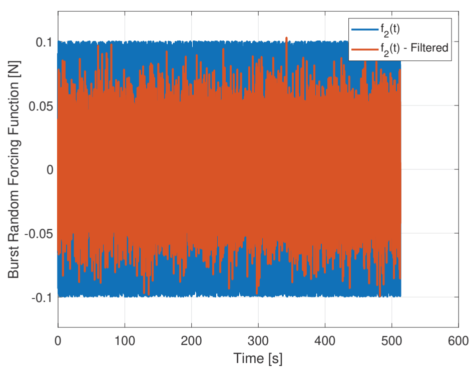
Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 1.023e+05$ (2.35%) $k_{3,cmplx} = 1.023e+05 + i1.808e+01$
H$_2$-NIXO		$k_3 = 1.039e+05$ (3.90%) $k_{3,cmplx} = 1.039e+05 + i2.341e+02$
H$_1$-NIXO with lin. data provided	×	$k_3 = 9.386e+04$ (6.14%) $k_{3,cmplx} = 9.386e+04 + i6.090e+03$
H$_2$-NIXO with lin. data provided	×	$k_3 = 9.926e+04$ (0.74%) $k_{3,cmplx} = 9.926e+04 + i5.261e+03$

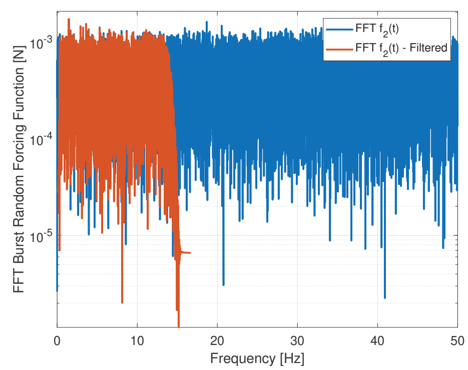
Tab. 81: SCENARIO B210: Input/Output Signals

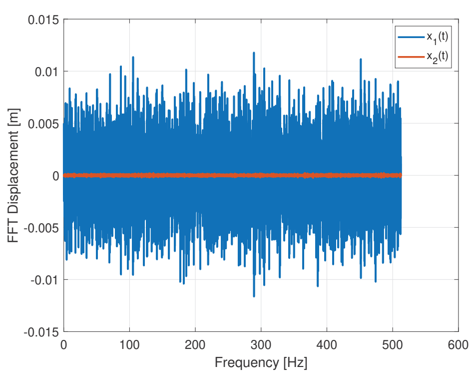
Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B210	112	5.000	0.100	3	8	0.3	15	0.3	15

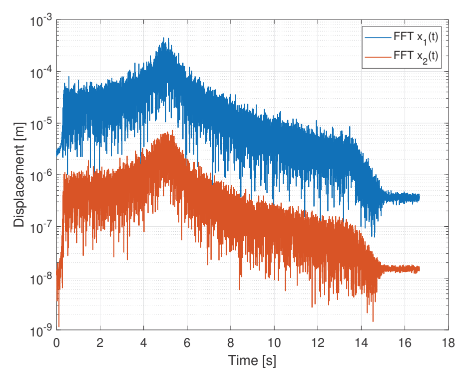
<table>
<tr><th>Signal</th><th>Time domain</th><th>Freq. domain</th></tr>
<tr><td>$f_I(t)$</td><td>


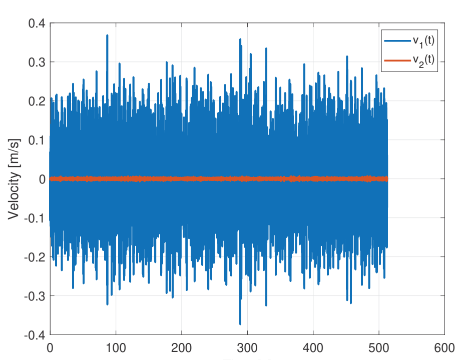
</td><td>


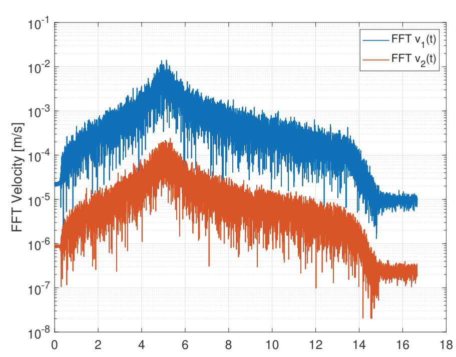
</td></tr>
<tr><td>$f_{II}(t)$</td><td>


</td><td>


</td></tr>
<tr><td>$x(t)$</td><td>


</td><td>


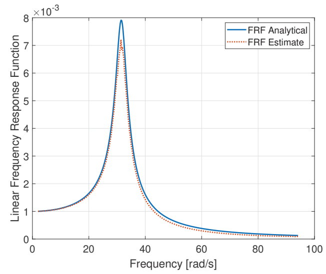
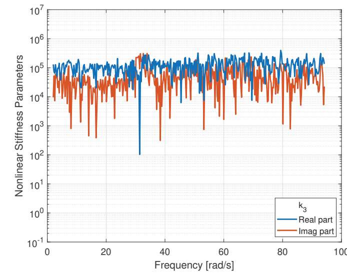
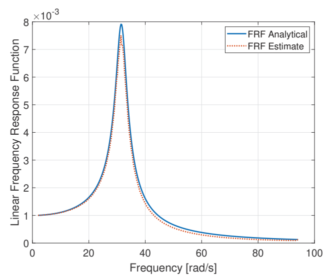
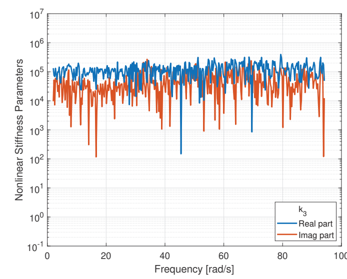
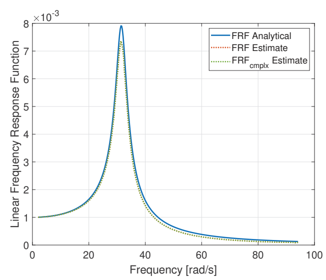
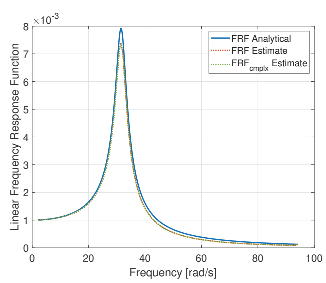
</td></tr>
<tr><td>$v(t)$</td><td>


</td><td>


</td></tr>
</table>

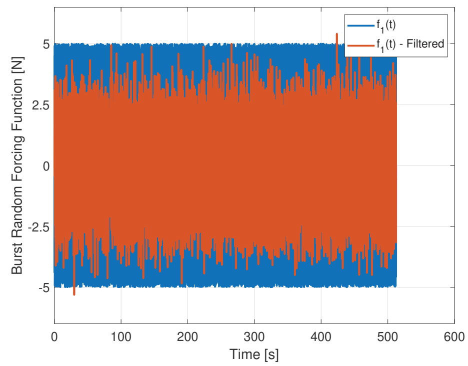
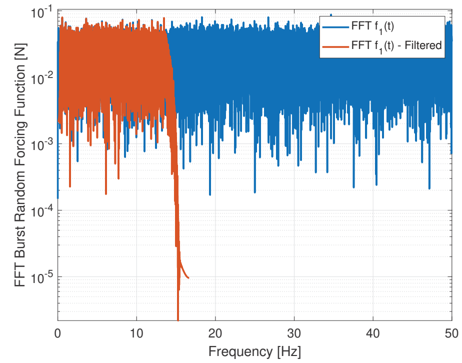
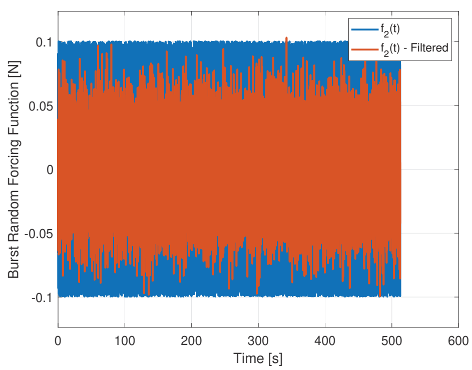
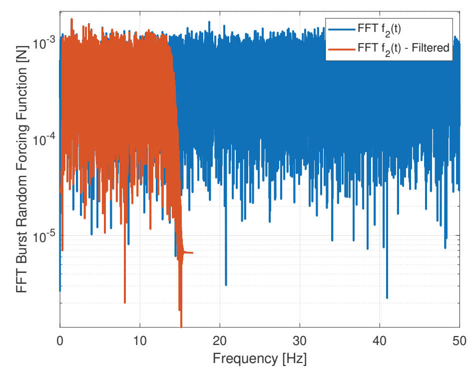
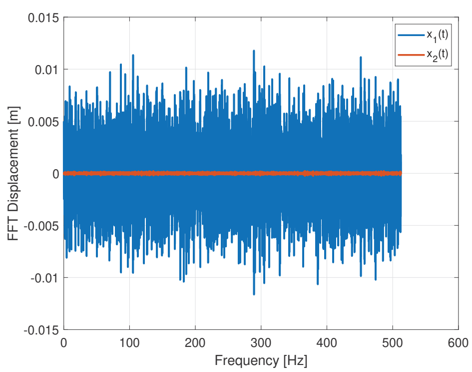
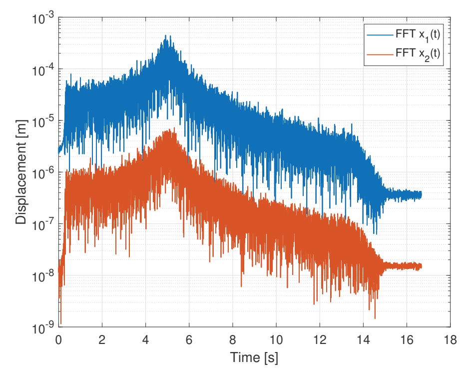
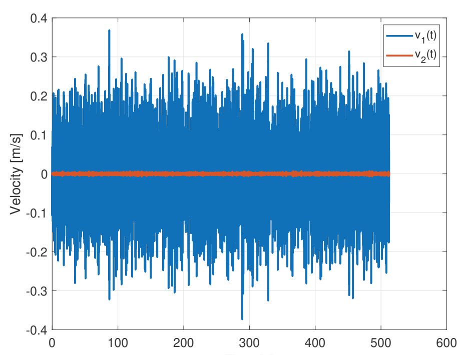
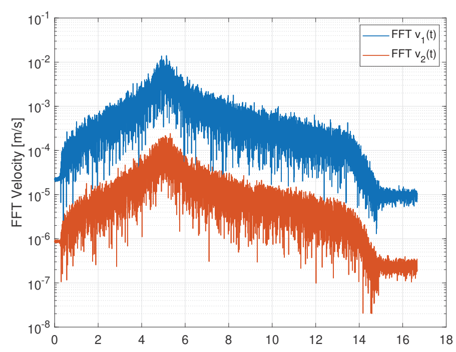
Tab. 82: SCENARIO B210: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B210	112	5.000	0.100	3	8	0.3	15	0.3	15

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 1.034e+05$ (3.38%) $k_{3,cmplx} = 1.034e+05 + i9.692e+01$
H$_2$-NIXO		$k_3 = 1.120e+05$ (12.00%) $k_{3,cmplx} = 1.120e+05 - i1.929e+04$
H$_1$-NIXO with lin. data provided	×	$k_3 = -2.055e+05$ (305.51%) $k_{3,cmplx} = -2.055e+05 + i3.576e+05$
H$_2$-NIXO with lin. data provided	×	$k_3 = 1.406e+05$ (40.56%) $k_{3,cmplx} = 1.406e+05 + i3.358e+05$

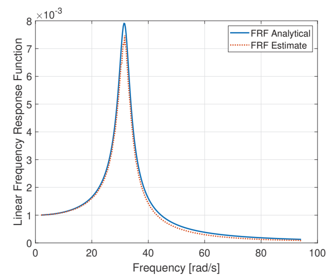
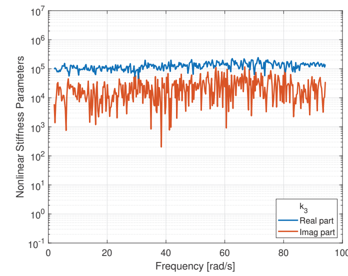
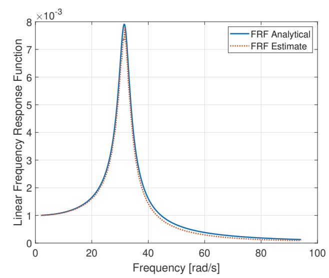
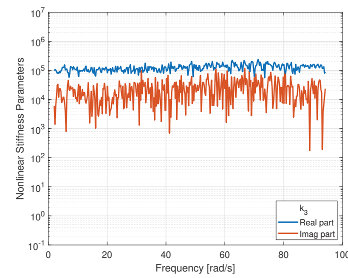
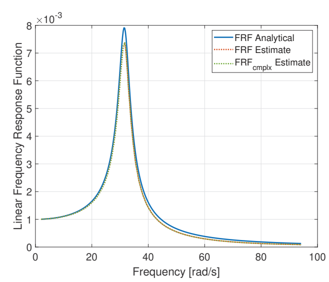
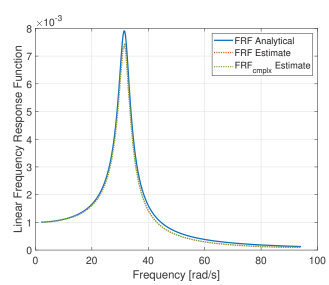
Tab. 83: SCENARIO B211: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B211	112	7.500	0.100	3	8	0.3	15	0.3	15

Signal	Time domain	Freq. domain
$f_I(t)$		
$f_{II}(t)$		
$x(t)$		
$v(t)$		

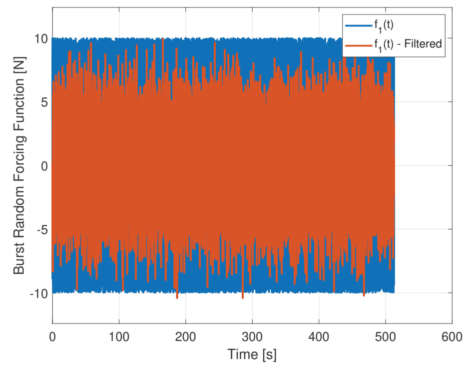
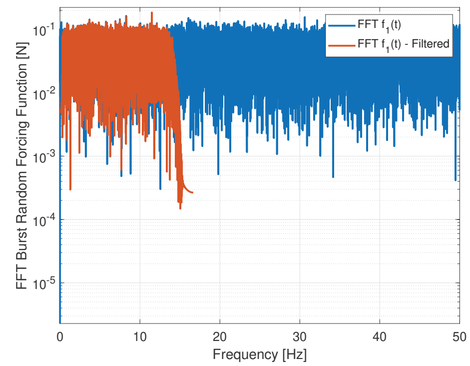
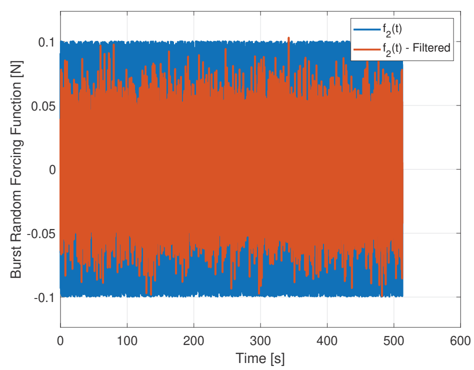
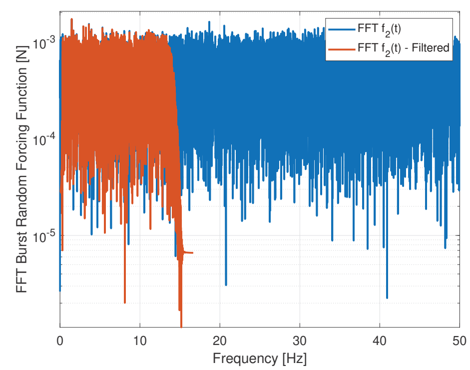
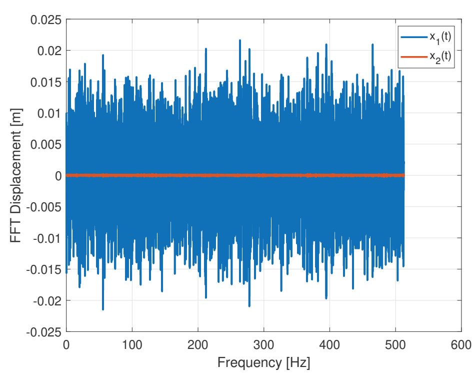
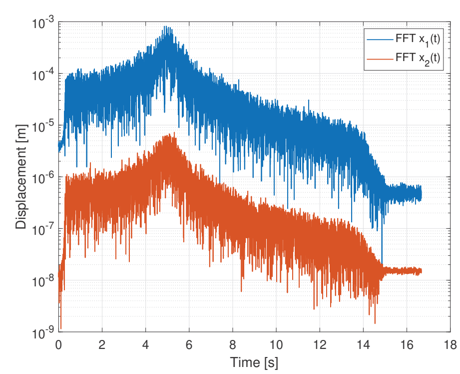
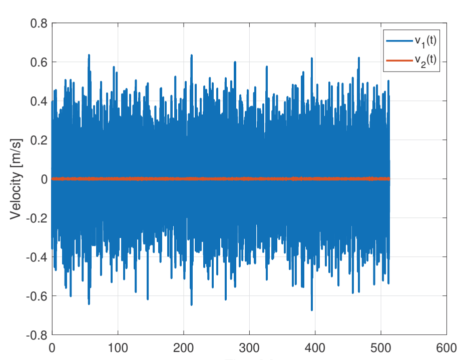
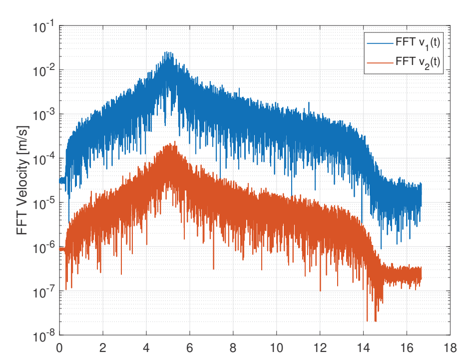
Tab. 84: SCENARIO B211: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B211	112	7.500	0.100	3	8	0.3	15	0.3	15

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 1.062e+05$ (6.20%) $k_{3,cmplx} = 1.062e+05 + i4.902e+03$
H$_2$-NIXO		$k_3 = 1.129e+05$ (12.85%) $k_{3,cmplx} = 1.129e+05 + i1.377e+04$
H$_1$-NIXO with lin. data provided	$\times$	$k_3 = -3.097e+04$ (130.97%) $k_{3,cmplx} = -3.097e+04 + i1.684e+05$
H$_2$-NIXO with lin. data provided	$\times$	$k_3 = 1.124e+05$ (12.41%) $k_{3,cmplx} = 1.124e+05 + i1.548e+05$

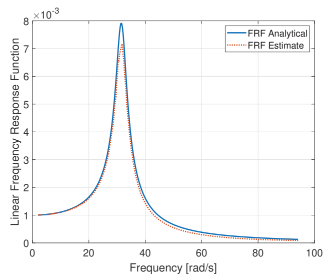
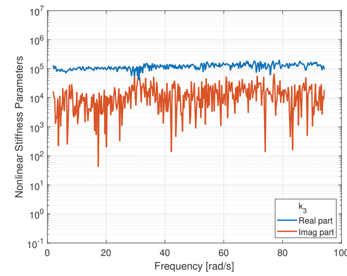
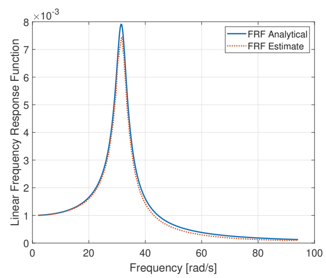
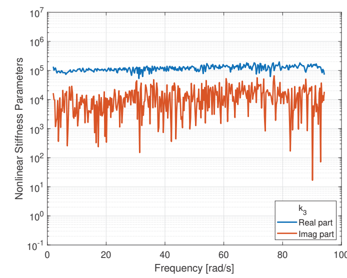
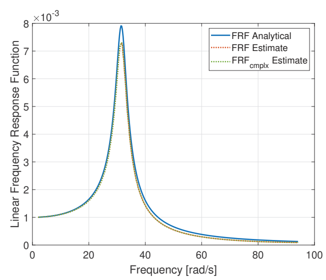
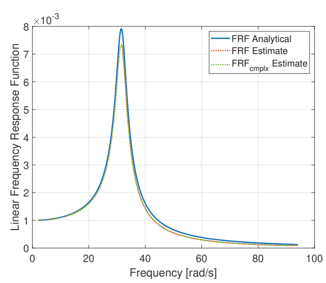
Tab. 85: SCENARIO B212: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B212	112	10.000	0.100	3	8	0.3	15	0.3	15

Signal	Time domain	Freq. domain
$f_I(t)$		
$f_{II}(t)$		
$x(t)$		
$v(t)$		

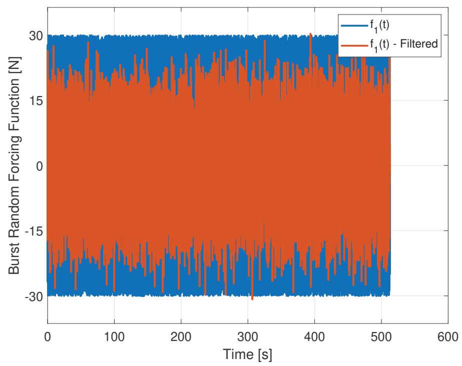
Tab. 86: SCENARIO B212: Results

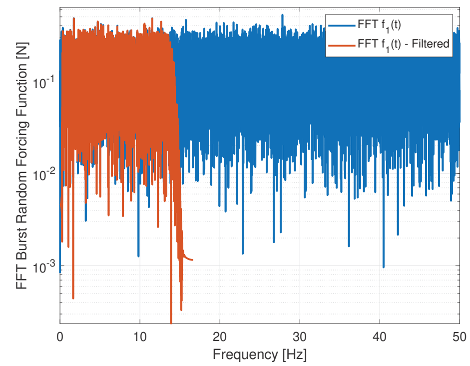
Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B212	112	10.000	0.100	3	8	0.3	15	0.3	15

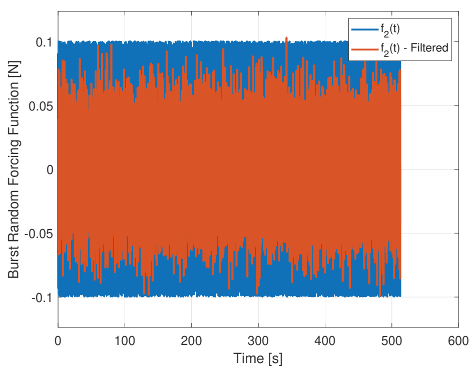
Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 1.073e+05$ (7.30%) $k_{3,cmplx} = 1.073e+05 + i2.951e+03$
H$_2$-NIXO		$k_3 = 1.084e+05$ (8.43%) $k_{3,cmplx} = 1.084e+05 + i2.489e+03$
H$_1$-NIXO with lin. data provided	×	$k_3 = 1.287e+04$ (87.13%) $k_{3,cmplx} = 1.287e+04 + i9.999e+04$
H$_2$-NIXO with lin. data provided	×	$k_3 = 1.061e+05$ (6.06%) $k_{3,cmplx} = 1.061e+05 + i9.906e+04$

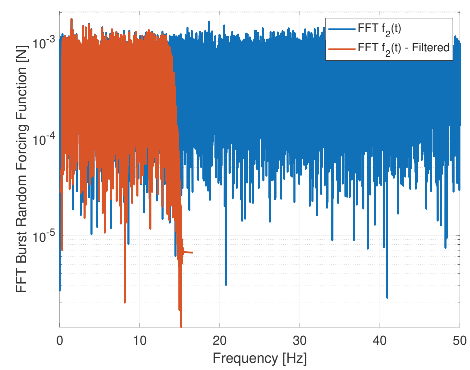
Tab. 87: SCENARIO B213: Input/Output Signals

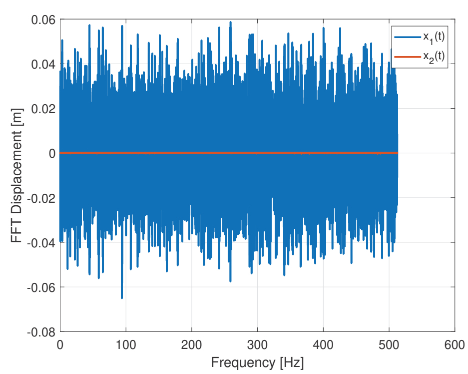
Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B213	112	30.000	0.100	3	8	0.3	15	0.3	15

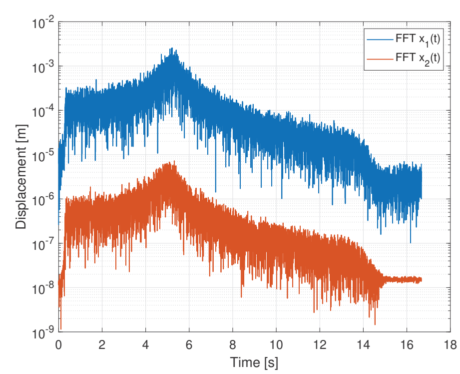
<table>
<tr><th>Signal</th><th>Time domain</th><th>Freq. domain</th></tr>
<tr><td>$f_I(t)$</td><td>


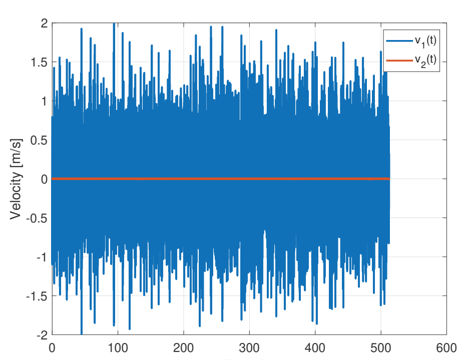
</td><td>


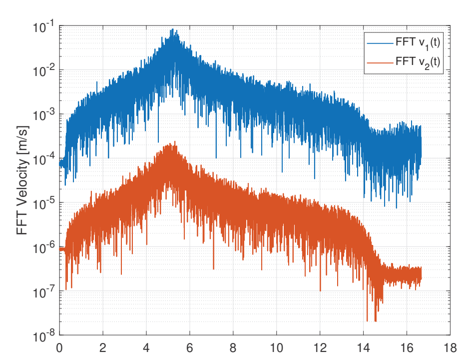
</td></tr>
<tr><td>$f_{II}(t)$</td><td>


</td><td>


</td></tr>
<tr><td>$x(t)$</td><td>


</td><td>


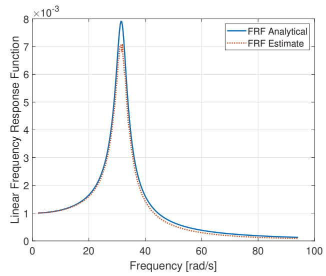
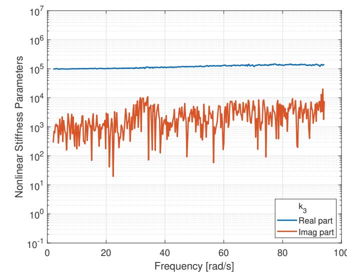
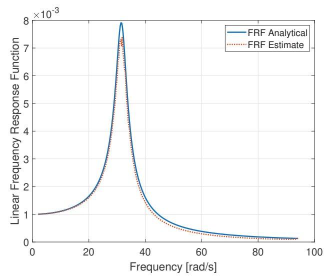
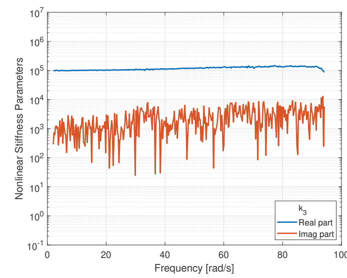
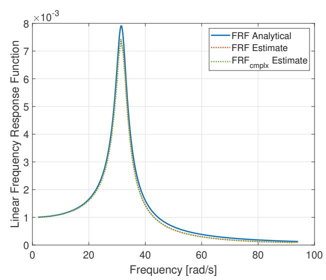
</td></tr>
<tr><td>$v(t)$</td><td>


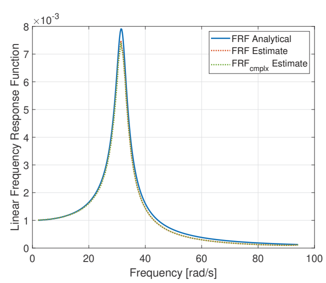
</td><td>


</td></tr>
</table>

Tab. 88: SCENARIO B213: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B213	112	30.000	0.100	3	8	0.3	15	0.3	15

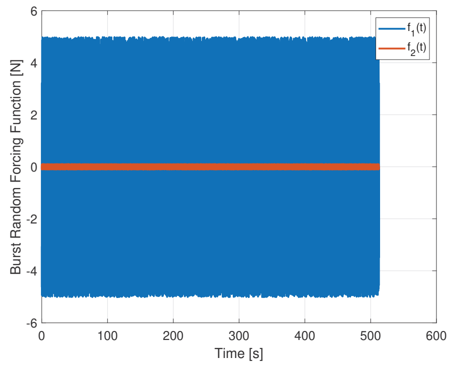
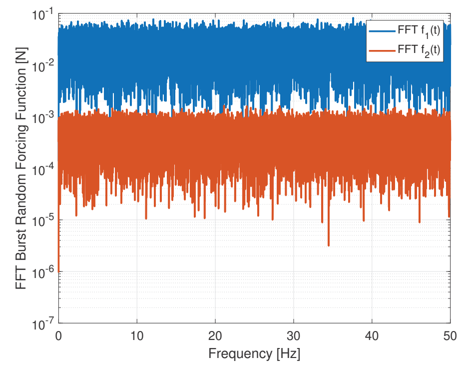
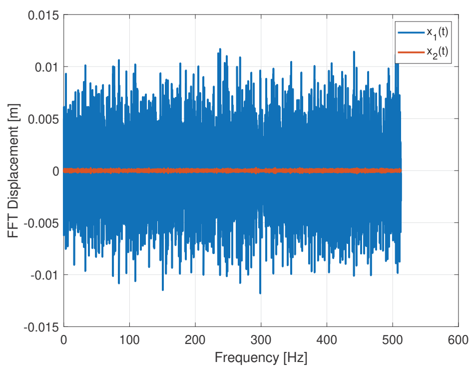
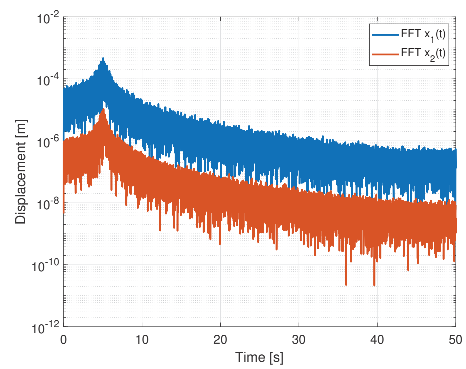
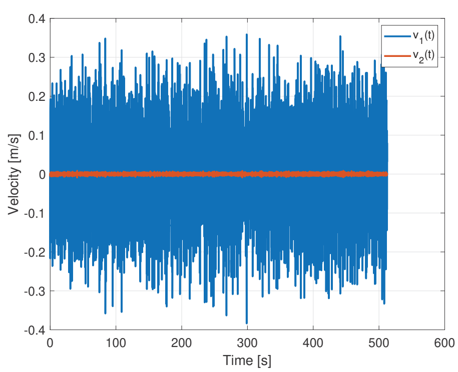
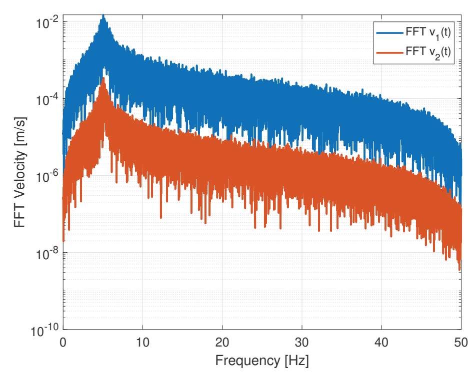
<table>
<tr><th>Method</th><th>Linear FRF estimate</th><th>Nonlin. param. estimates
(rel. error)</th></tr>
<tr><td>H$_1$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_2$-NIFO</td><td>
</td><td>
</td></tr>
<tr><td>H$_1$-NIXO</td><td>
</td><td>$k_3 = 1.070e+05$ (6.96%)

$k_{3,cmplx} = 1.070e+05 \; - \; i1.696e+02$</td></tr>
<tr><td>H$_2$-NIXO</td><td>
</td><td>$k_3 = 1.083e+05$ (8.35%)

$k_{3,cmplx} = 1.083e+05 \; - \; i1.486e+03$</td></tr>
<tr><td>H$_1$-NIXO
with lin. data
provided</td><td>×</td><td>$k_3 = 8.844e+04$ (11.56%)
$k_{3,cmplx} = 8.844e+04 \; + \; i1.305e+04$</td></tr>
<tr><td>H$_2$-NIXO
with lin. data
provided</td><td>×</td><td>$k_3 = 9.901e+04$ (0.99%)
$k_{3,cmplx} = 9.901e+04 \; + \; i1.224e+04$</td></tr>
</table>

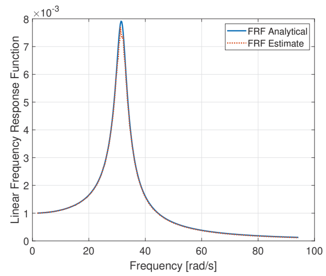
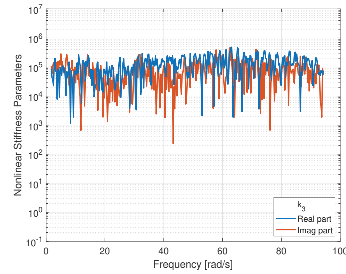
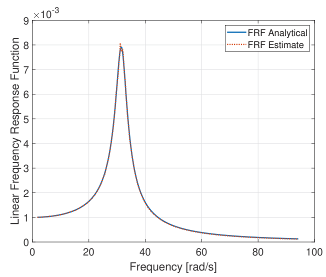
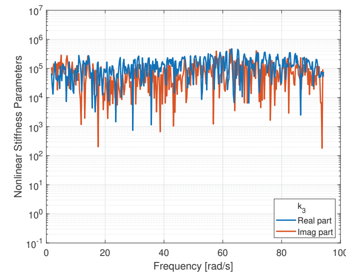
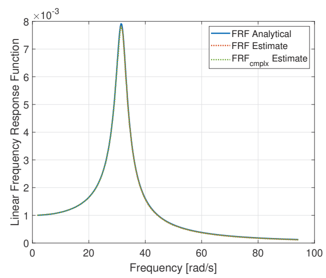
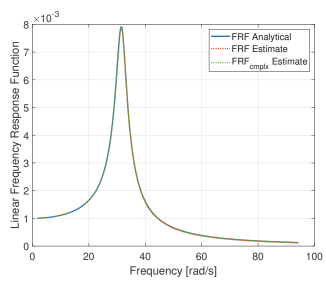
Tab. 89: SCENARIO B214: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B214	113	5.000	0.100	-	-	-	-	0.3	15

Signal	Time domain	Freq. domain
$f_I(t)$ and $f_{II}(t)$		
$x(t)$		
$v(t)$		

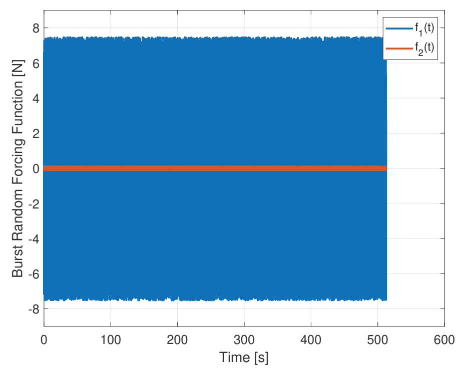
Tab. 90: SCENARIO B214: Results

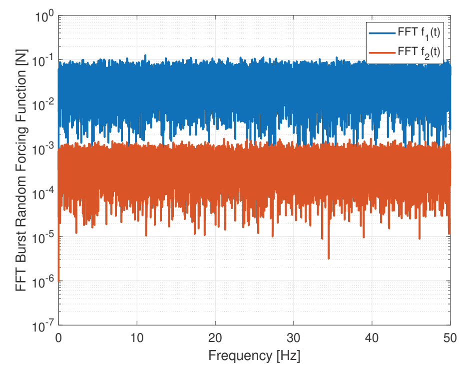
Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B214	113	5.000	0.100	-	-	-	-	0.3	15

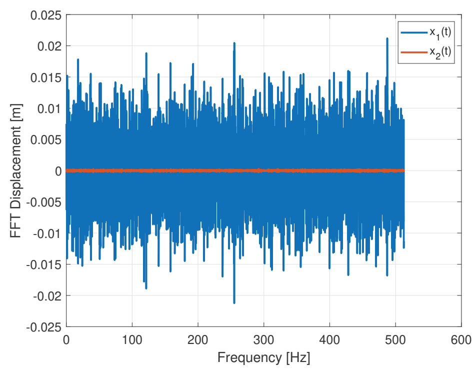
Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 1.087e+05$ (8.65%) $k_{3,cmplx} = 1.087e+05 - i7.294e+03$
H$_2$-NIXO		$k_3 = 1.046e+05$ (4.56%) $k_{3,cmplx} = 1.046e+05 + i1.016e+04$
H$_1$-NIXO with lin. data provided	$\times$	$k_3 = 5.908e+04$ (40.92%) $k_{3,cmplx} = 5.908e+04 + i5.534e+04$
H$_2$-NIXO with lin. data provided	$\times$	$k_3 = 1.066e+05$ (6.56%) $k_{3,cmplx} = 1.066e+05 + i3.719e+04$

Tab. 91: SCENARIO B215: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B215	113	7.500	0.100	-	-	-	-	0.3	15

<table>
<tr><th>Signal</th><th>Time domain</th><th>Freq. domain</th></tr>
<tr><td>$f_I(t)$ and $f_{II}(t)$</td><td>


</td><td>


</td></tr>
<tr><td>$x(t)$</td><td>


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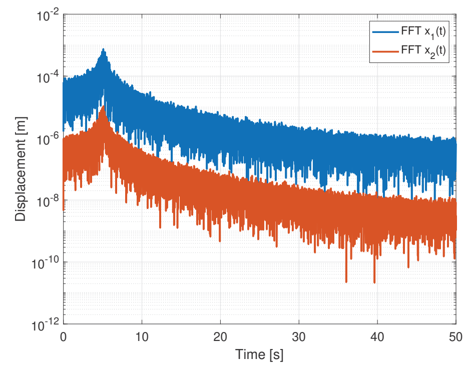
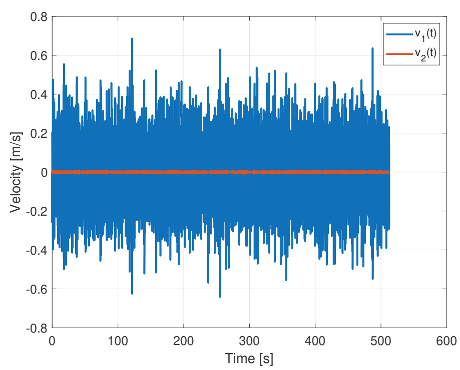


</td></tr>
<tr><td>$v(t)$</td><td>


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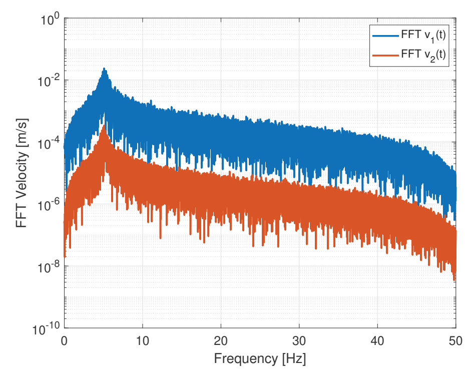
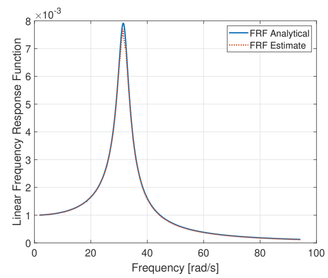
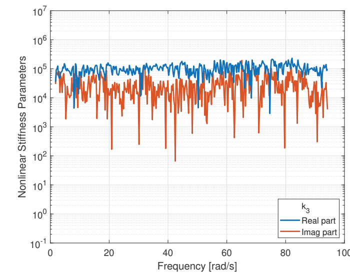
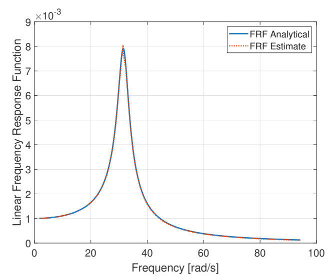
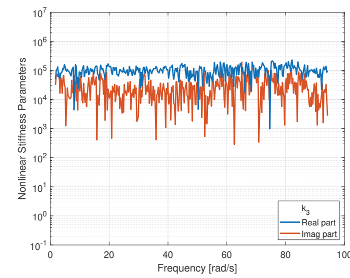
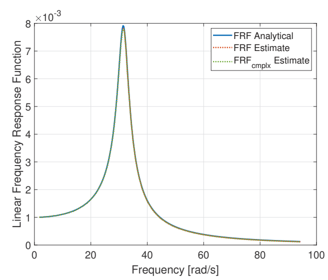
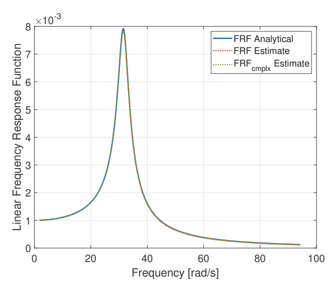


</td></tr>
</table>

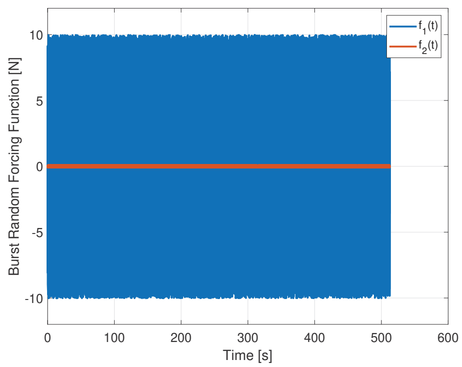
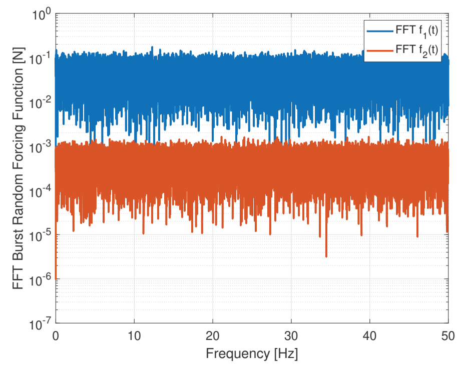
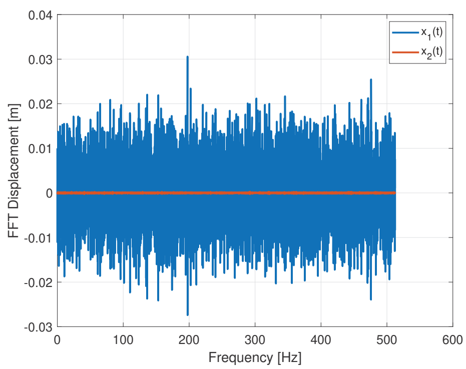
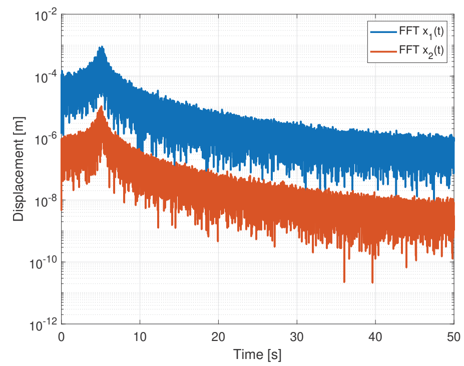
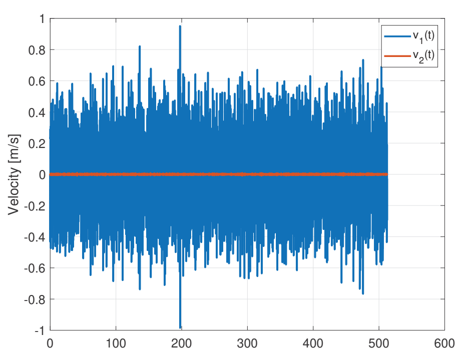
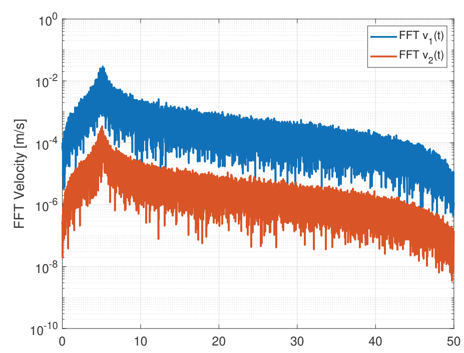
Tab. 92: SCENARIO B215: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B215	113	7.500	0.100	-	-	-	-	0.3	15

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 1.004e+05$ (0.40%) $k_{3,cmplx} = 1.004e+05 - i5.631e+03$
H$_2$-NIXO		$k_3 = 9.651e+04$ (3.49%) $k_{3,cmplx} = 9.651e+04 - i1.206e+04$
H$_1$-NIXO with lin. data provided	$\times$	$k_3 = 7.930e+04$ (20.70%) $k_{3,cmplx} = 7.930e+04 + i2.181e+04$
H$_2$-NIXO with lin. data provided	$\times$	$k_3 = 9.789e+04$ (2.11%) $k_{3,cmplx} = 9.789e+04 + i1.273e+04$

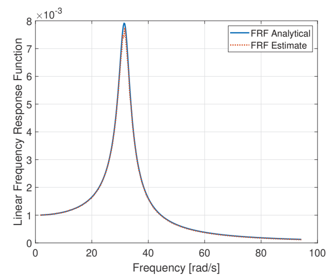
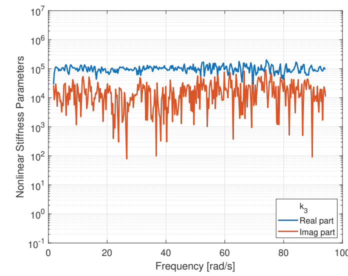
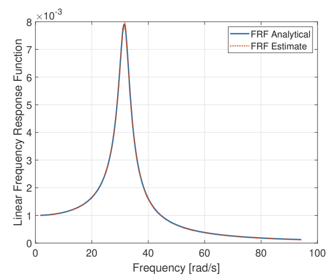
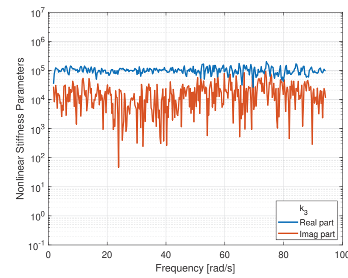
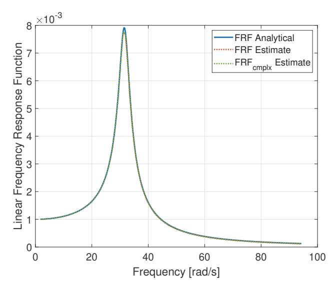
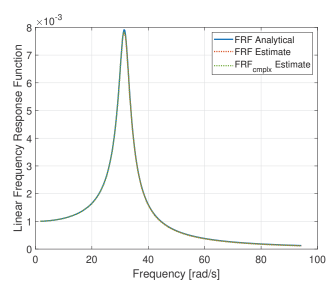
Tab. 93: SCENARIO B216: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B216	113	10.000	0.100	-	-	-	-	0.3	15

<table>
<tr><th>Signal</th><th>Time domain</th><th>Freq. domain</th></tr>
<tr><td>$f_I(t)$
and
$f_{II}(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$x(t)$</td><td>
</td><td>
</td></tr>
<tr><td>$v(t)$</td><td>
</td><td>
</td></tr>
</table>

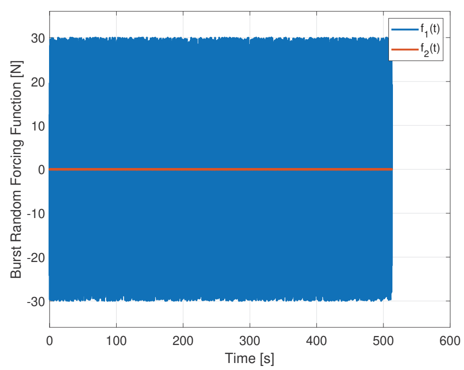
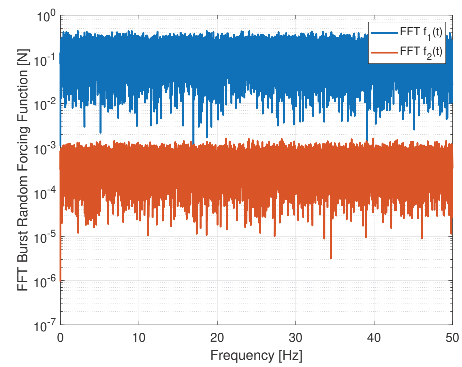
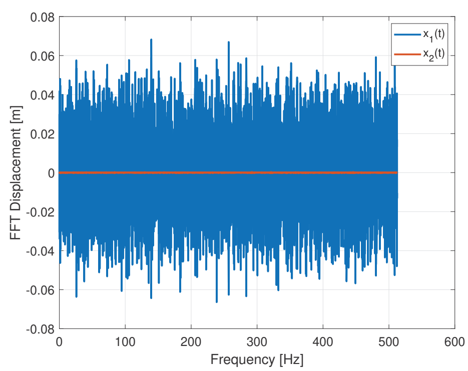
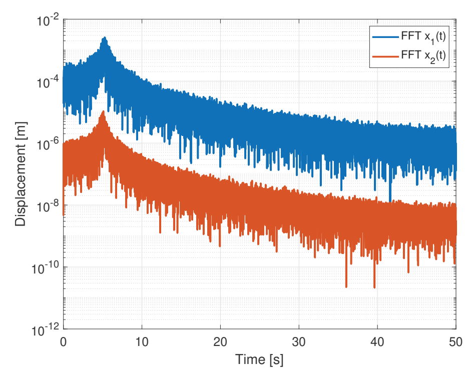
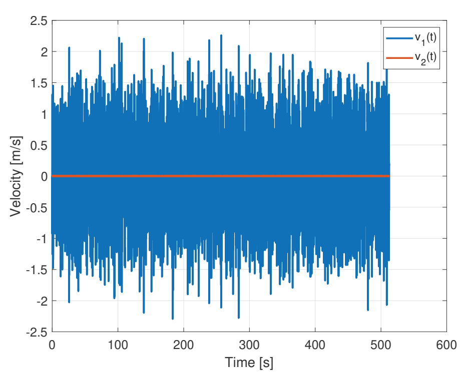
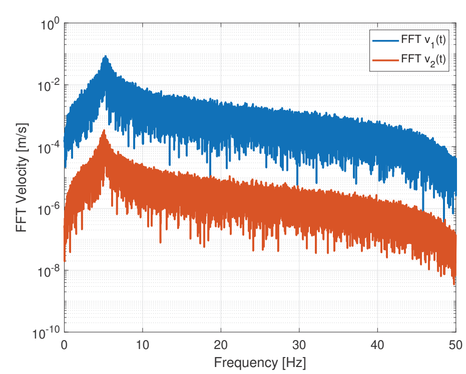
Tab. 94: SCENARIO B216: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B216	113	10.000	0.100	-	-	-	-	0.3	15

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 1.025e+05$ (2.48%) $k_{3,cmplx} = 1.025e+05 \; + \; i1.348e+03$
H$_2$-NIXO		$k_3 = 9.999e+04$ (0.01%) $k_{3,cmplx} = 9.999e+04 \; + \; i4.056e+03$
H$_1$-NIXO with lin. data provided	×	$k_3 = 8.896e+04$ (11.04%) $k_{3,cmplx} = 8.896e+04 \; + \; i1.535e+04$
H$_2$-NIXO with lin. data provided	×	$k_3 = 9.864e+04$ (1.36%) $k_{3,cmplx} = 9.864e+04 \; + \; i1.172e+04$

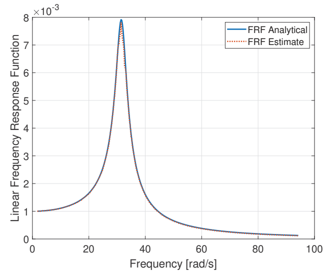
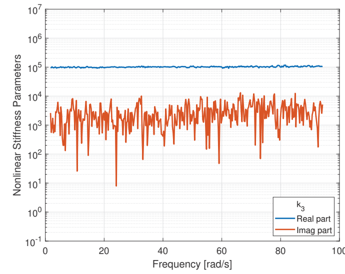
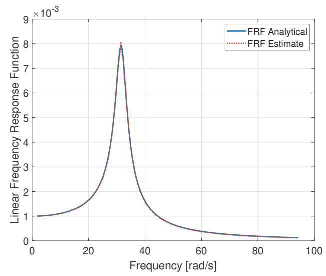
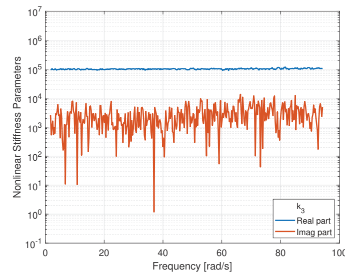
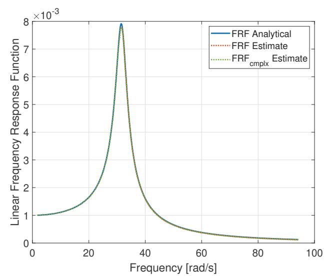
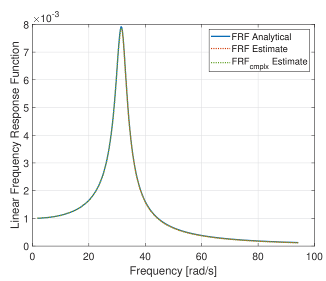
Tab. 95: SCENARIO B217: Input/Output Signals

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B217	113	30.000	0.100	-	-	-	-	0.3	15

Signal	Time domain	Freq. domain
$f_I(t)$ and $f_{II}(t)$		
$x(t)$		
$v(t)$		

Tab. 96: SCENARIO B217: Results

Scenario	Signal	F_I or F	F_{II}	DF	FO	Ω_1^{filt}	Ω_2^{filt}	Ω_1^{spect}	Ω_2^{spect}
B217	113	30.000	0.100	-	-	-	-	0.3	15

Method	Linear FRF estimate	Nonlin. param. estimates (rel. error)
H$_1$-NIFO		
H$_2$-NIFO		
H$_1$-NIXO		$k_3 = 1.006e+05$ (0.59%) $k_{3,cmplx} = 1.006e+05 - i2.308e+02$
H$_2$-NIXO		$k_3 = 1.011e+05$ (1.08%) $k_{3,cmplx} = 1.011e+05 + i5.144e+02$
H$_1$-NIXO with lin. data provided	$\times$	$k_3 = 9.830e+04$ (1.70%) $k_{3,cmplx} = 9.830e+04 + i1.850e+03$
H$_2$-NIXO with lin. data provided	$\times$	$k_3 = 9.993e+04$ (0.07%) $k_{3,cmplx} = 9.993e+04 + i1.173e+03$

Comments

1. Comments to this section are very similar to those on page 55, with the caveat that the decimation factor does not affect results significantly.

References

[1] Adams, D., and Allemang, R., 2000. "A frequency domain method for estimating the parameters of a non-linear structural dynamic model through feedback". *Mechanical Systems and Signal Processing,* **14**(4), pp. 637 – 656.

[2] Haroon, M., and Adams, D. E., 2009. "A modified H_2 algorithm for improved frequency response function and nonlinear parameter estimation". *Journal of Sound and Vibration,* **320**, 03, pp. 822–837.