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FLOQUET EXPERIMENTAL MODAL ANALYSIS FOR SYSTEM IDENTIFICATION OF LINEAR TIME-PERIODIC SYSTEMS

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ABSTRACT

A variety of systems can be faithfully modeled as linear with coefficients that vary periodically with time or Linear Time-Periodic (LTP). Examples include anisotropic rotor-bearing systems, wind turbines, satellite systems, etc… A number of powerful techniques have been presented in the past few decades, so that one might expect to model or control an LTP system with relative ease compared to time varying systems in general. However, few, if any, methods exist for experimentally characterizing LTP systems. This work seeks to produce a set of tools that can be used to characterize LTP systems completely through experiment. While such an approach is commonplace for LTI systems, all current methods for time varying systems require either that the system parameters vary slowly with time or else simply identify a few parameters of a pre-defined model to response data. A previous work presented two methods by which system identification techniques for linear time invariant (LTI) systems could be used to identify a response model for an LTP system from free response data. One of these allows the system's model order to be determined exactly as if the system were linear time-invariant. This work presents a means whereby the response model identified in the previous work can be used to generate the full state transition matrix and the underlying time varying state matrix from an identified LTP response model and illustrates the entire system-identification process using simulated response data for a Jeffcott rotor in anisotropic bearings.

1. INTRODUCTION

A number of important dynamic systems can be modeled as Linear Time-Periodic (LTP). When this is the case, it is exceedingly important for the analyst to detect and accurately model this character since it can lead to instability and resonance at frequencies other than those predicted by theory for Linear Time-Invariant (LTI) systems. Floquet and Lyapunov developed some important theories regarding linear differential equations with periodic coefficients in the late 1800's [1], so the theory of time-periodic systems is usually called Floquet theory or Floquet-Lyapunov theory. This theory has been applied to a variety of mechanical systems such as helicopters, wind turbines or other bladed machines [2], [3], mechanisms [4], buckling problems [2], satellites and rotating machinery [5], [6], [7], [8]. In his text, Richards [9] gives a detailed history of Floquet theory, and reviews applications of second order LTP systems, including: mass spectrometry, dynamic buckling of structures, elliptical waveguides, and electronics. Sinha and his colleagues [10] have published a number of works on the analysis and control of linear and nonlinear time-periodic systems. Montagnier, Spiteri and Angeles [4] recently presented a thorough review of Floquet theory.

Despite these advances in the analytical realm, little progress has been reported regarding experimental characterization of LTP systems. While methods for identifying parametric models of LTI systems are well established [11] [12], and a multitude of methods for identifying nonlinear systems have been presented [13], the same can hardly be said for LTP systems. In [14], the author and Ginsberg took a first step in this direction and presented two methods by which the free response of an LTP system can be parameterized using standard tools for LTI systems and without the need to guess at the system's model order a priori. Tremendous synergy was noted between the free response of LTP and LTI systems, so one would expect that an LTP system can be characterized almost as easily as an LTI system using the proposed methods. This work could have significant implications in a number of applications where LTP models are either difficult to derive analytically or difficult to validate.

This work expands upon the previous, demonstrating how the system's state transition matrix can be identified from the Fourier series response model presented in [14]. Furthermore, this work shows that the time varying system matrix can be recovered from the state transition matrix if enough of the states are measured.

The following section reviews Floquet-Lyapunov theory and summarizes two strategies for identifying the parameters of LTP systems from free response data. The new methods for finding the state transition matrix and the system matrix from the identified models are then presented. These methods are applied to a system whose parameters vary with time, a simple model of a Jeffcott rotor on an anisotropic shaft and anisotropic bearings. Section 4 presents some conclusions.

2. THEORETICAL DEVELOPMENT

2.1. Floquet-Lyapunov Theory for LTP Systems

This section presents a brief review of some concepts from Floquet Theory. The equations governing a linear time varying system may be written in the following state space representation

$$\dot{x} = A(t)x + B(t)u \tag{1}$$

where x is the system state vector, $\{u\}$ the inputs to the system, and the matrices $A(t)$ and $B(t)$ vary with time. This work considers only the case where the input $\{u\} = 0$, in which case the state transition matrix can be used to transfer the state vector from the initial state $x(t_0)$ at time t_0 to the state at time t as follows

$$x(t) = \Phi(t,t_0)x(t_0) \qquad (2).$$

The semi-group property follows immediately from this definition.

$$\Phi(t,t_0) = \Phi(t,t_1)\Phi(t_1,t_0) \tag{3}$$

If $A(t)$ is periodic with period T_A, then the state transition matrix for $t \geq 0$ can be reconstructed from the state transition matrix for $0 \leq t < T_A$ as follows [4]

$$\Phi(t+nT_A,t_0) = \Phi(t,t_0)\Phi(t_0+T_A,t_0)^n \tag{4}$$

where n is an integer. This important result is often exploited to efficiently compute the response of LTP systems [2].

The Floquet-Lyapunov theorem states that the state transition matrix of an LTP system with period T_A can be decomposed as follows [2]

$$\Phi(t,t_0) = P(t)^{-1}\exp\left(R(t-t_0)\right)P(t_0)\,, \tag{5}$$

where R is a constant matrix and $P(t)$ is periodic such that $P(t+T_A)=P(t)$. Both matrices can be complex in general [4]. It was noted in [14], that if R can be diagonalized

$$R = (M_R)\Lambda_R(M_R)^{-1}\,, \tag{6}$$

then the Floquet-Lyapunov representation can be expressed in terms of a set of complex modal parameters with time varying mode shapes $\{\psi_r\}$ as follows

$$\begin{gathered}\Phi(t,t_0) = \Psi(t)\exp\left(\Lambda_R(t-t_0)\right)\Psi(t_0)^{-1} \\ \Psi(t) = P(t)^{-1}M_R\end{gathered} \qquad (7),$$

where $\Psi(t)=[\{\psi_1\}, \{\psi_2\}, \ldots]$ is the time varying modal matrix and Λ_R contains the constant Floquet eigenvalues of the system.

2.2. Identifying Models of LTP Systems

2.2.1. Multiple Discrete-Time Systems (MDTS) Method

Two methods for identifying the parameters of LTP systems were presented in [14]. The first, later dubbed the Multiple Discrete-Time Systems (MDTS) method, notes that the system in eq. (7) can be treated as a discrete-time LTI system if the sample increment is $p*T_A$ where p is an integer and $p \geq 1$. Alternatively, one can probe the time-varying nature of the system using a sample increment of T_A/p. In this case one can create an array of LTI responses that reproduce the measured response exactly. Each LTI response has a sample increment of T_A, each starts at a different initial time, and the initial times span $[0, T_A)$. The collection then characterizes the time varying nature of the system. A derivation of the MDTS method, based on that which was first presented in [14], is included in Section 1 of the Appendix for convenience.

2.2.2. Fourier Series Expansion (FSE) Method

The other approach, later dubbed the Fourier Series Expansion (FSE) approach, is derived by considering the summation form of eq. (7).

$$\begin{gathered}\Phi(t,t_0) = \sum_{r=1}^{2N}\left(A_R\right)_r \exp\left(\lambda_r(t-t_0)\right) \\ \left(A_R\right)_r = \left\{\psi_r(t)\right\}\left\{L_r(t_0)\right\}^T\end{gathered} \qquad (8),$$

where N denotes the number of modes, $\psi_r(t)$ are the mode vectors or the columns of $[\Psi(t)]$, and $L_r(t)$ are the modal participation factors, or the columns of $[\Psi(t)]^{-T}$. The residue matrices $(A_R)_r$ are the product of a periodic column vector and a constant row vector, and hence are themselves periodic. Because they are periodic, they can be readily expanded in a Fourier series. They are assumed to be adequately represented using a fixed number $(2*N_R + 1)$ of terms so that the response may be written as follows.

$$\Phi(t,t_0) = \sum_{r=1}^{2N}\sum_{m=-N_R}^{N_R}\left[B\right]_{r,m}\exp\left(\left(\lambda_r + im\omega_A\right)(t-t_0)\right) \tag{9}$$

This is simply the free response of an LTI system with $2*N*(2*N_R+1)$ eigenvalues. The LTP system actually has only $2*N$ eigenvalues, but the Fourier series coefficients of the residue matrices, which are composed of the time-varying mode vectors) modulate the response so that it appears that the system has $2*N*(2*N_R)$ additional eigenvalues. Hence, the free response can be parameterized using an LTI system identification routine, following which one can collect the replicates of each eigenvalue and their associated coefficients.

While this approach is certainly possible, the author has found it much more convenient to use the MDTS method whenever the sample increment allows because it yields Frequency Response Functions (FRFs) that are easier to interpret because the model order is readily apparent. However, the FSE representation is extremely useful, as will be shown in the following section. In cases where an FSE model is needed, the author has found it more convenient to use the MDTS method to identify a model for the system and then to expand the time varying residue matrices found by the MDTS method into a Fourier series model.

2.3. Extensions

The previous work [14] discussed finding parametric models for the free response of an LTP system, yet it did not discuss how one would then go about determining the State Transition Matrix (STM), nor how one could determine the time-varying state matrix $A(t)$. This section addresses these two issues.

2.3.1. Reconstructing the State Transition Matrix using the identified response model

In practice, one is likely to measure the response $y(t)$ of an LTP system to an initial condition at a certain number N_o of outputs

$$y(t) = C\Phi(t,t_0)x(t_0) \qquad (10).$$

The output matrix C relates the outputs to the state vector $x(t)$. The measured response y(t) uniquely determines the state of the system if C is full rank, in which case

$$x(t) = C^{-1}y(t) \qquad (11).$$

If the system is a second-order structural dynamic system, defined by time varying mass, stiffness and damping matrices, then half of the states are simply the derivatives of the other half. Furthermore, in these cases one often has more output measurement locations than states, so an arbitrary but sufficient state vector can be constructed from the displacement responses by differentiating them. For the present development we shall assume that the proper number of outputs was measured and that they are the desired states of the system matrix, so C will be taken to be C = $[I_{NxN}, 0_{NxN}]$. The states measured are assumed to be those describing the displacement of the system, so the remaining states are their derivatives. The response is fit to the following LTP model where $\{B_y\}$ are the coefficients of a FSE of the residue vectors of the response and λ_r are its eigenvalues.

$$y(t) = \sum_{r=1}^{2N} \sum_{m=-N_R}^{N_R} \left\{B_y\right\}_{r,m} \exp\left(\left(\lambda_r + im\omega_A\right)(t-t_0)\right) \qquad (12)$$

The missing states can be found by differentiating the previous equation resulting in the following.

$$\begin{aligned}\dot{y}(t) = \sum_{r=1}^{2N} \sum_{m=-N_R}^{N_R} \left\{B_y\right\}_{r,m} \left(\lambda_r + im\omega_A\right) \times \\ \exp\left(\left(\lambda_r + im\omega_A\right)(t-t_0)\right)\end{aligned} \qquad (13)$$

A valid state vector can now be recovered as

$$x(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}, \qquad (14)$$

Assuming that C is N by N. One now has an FSE model for the following.

$$x(t) = \Phi(t,t_0)x(t_0) \qquad (15)$$

Both $x(t)$ and $x(t_0)$ in the previous equation are known using the models in eqs. (12) and (13), yet one would need $2N$ independent pairs of responses and initial conditions to uniquely determine the STM. Fortunately, one can generate the required responses by shifting the response in eq. (15) by nT_A, where n is an integer, and using the periodicity of the STM.

$$\begin{aligned} x_n\left(t + nT_A\right) &= \Phi\left(t + nT_A, t_0 + nT_A\right) x_n\left(t_0 + nT_A\right) \\ &= \Phi\left(t, t_0\right) x_n\left(t_0 + nT_A\right) \\ n &= 0,1,\ldots\left(2N-1\right) \end{aligned} \qquad (16)$$

One can use eqs. (12) and (13) to show that each of the responses in eq. (16) has the following form,

$$\begin{aligned} x_n(t + nT_A) = \sum_{r=1}^{2N} \sum_{m=-N_R}^{N_R} \begin{bmatrix} \left\{B_y\right\}_{r,m} \\ \left\{B_y\right\}_{r,m} \left(\lambda_r + im\omega_A\right) \end{bmatrix} \times \\ \exp\left(nT_A\lambda_r\right) \exp\left(\left(\lambda_r + im\omega_A\right)(t-t_0)\right) \end{aligned} \qquad (17),$$

which was simplified by noting that $\exp(im\omega_A nT_A) = 1$.

The STM is then found by solving the following linear system of equations.

$$\begin{aligned} \left[x_0(t) \quad \cdots \quad x_{2N-1}(t + \left(2N-1\right)T_A)\right] = \\ \Phi\left(t,t_0\right)\left[x_0\left(t_0\right) \quad \cdots \quad x_{(2N-1)}\left(t_0 + \left(2N-1\right)T_A\right)\right] \end{aligned} \qquad (18)$$

Substituting eq. (17) into the matrix on the left hand side above, eq. (9) for the STM above, and then matching terms in the summations, it becomes apparent that one can take

$$\{\psi_r(t)\} = \sum_{m=-N_R}^{N_R} \begin{bmatrix} \{B_y\}_{r,m} \\ \{B_y\}_{r,m}(\lambda_r + im\omega_A) \end{bmatrix} \exp\left(im\omega_A(t-t_0)\right) \tag{19}$$

(19),

and then evaluate at t_0 to find $\{L_r(t_0)\}$.

2.3.2. Reconstructing the Time Varying System Matrix

Once a FSE model for the State Transition Matrix has been identified, one can use the STM differential equation:

$$\frac{d}{dt}\Phi(t,t_0) = A(t)\Phi(t,t_0) \tag{20}$$

to solve for the time varying system matrix $A(t)$ as follows.

$$A(t) = \left(\frac{d}{dt}\Phi(t,t_0)\right)\Phi(t,t_0)^{-1} \tag{21}$$

This was accomplished numerically in the applications that follow at a number of time steps t in the interval $[0,T_A)$. The derivative of the STM was found by differentiating the FSE, analogous to what was done in eq. (13).

3. APPLICATION

3.1. Response Model Identification

The proposed system identification methods will be demonstrated by applying them to simulated data from an LTP system. The system represents a Jeffcott rotor on an anisotropic shaft that is supported by anisotropic bearings. A lumped parameter model of this system is shown in Figure 2, which consists of a point mass suspended by two orthogonal, massless springs with spring constants k_{Rx} and k_{Ry}. The springs are attached to a massless turntable that turns at constant speed Ω. The turntable is fixed to ground by two massless springs k_{Fx} and k_{Fy}. The equations of motion for this system are given in the Appendix. Stiffness proportional damping is added to the system via the factor c_f. The parameters used in this example are: $k_{Rx} = 1$, $k_{Ry} = 1.2$, $c_f = 0.004$, $k_{Fx} = 1$, $k_{Fy} = 1.5$, $m = 1$, $\Omega = 0.5$. This represents a system whose parameters vary significantly with time; the natural frequencies of the system with the shaft held fixed at various angles ranged from 0.707 and 0.911 rad/s. Stability analysis, performed by computing the state transition matrix (STM) at the end of one cycle using numerical integration and then finding the Floquet multipliers of the STM, reveals that the system is unstable for $0.73 < \Omega < 0.90$. The system is always stable if $k_{Rx} = k_{Ry}$. It would certainly be important to detect and properly characterize the time-varying nature of this system if it is to operate at high speeds.

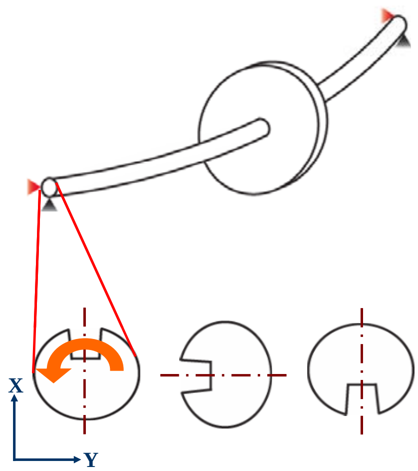


Figure 1: Anisotropic shaft, due to a keyway, on anisotropic bearings. This system can be modeled as Linear Time-Periodic (LTP).

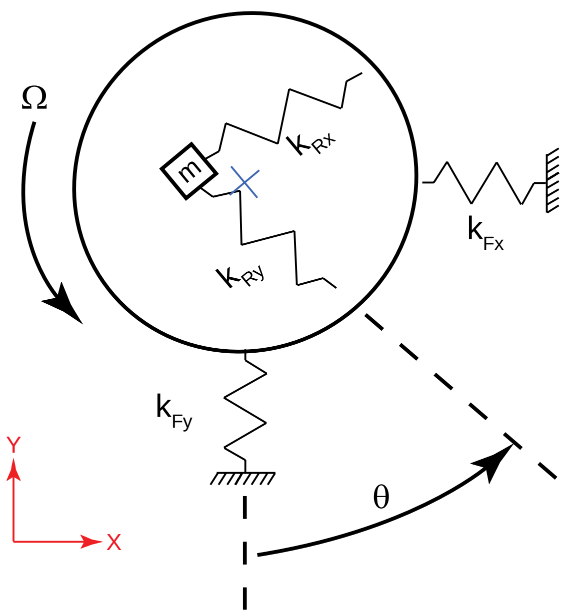


Figure 2: Schematic of Simple LTP System

The response of the system to a unit impulsive force, administered at an angle of 45 degrees from the x axis, was found. This is equivalent to the free response of the system to an initial velocity in the direction of the force. Specifically, the initial state vector is $x_0 = [0, 0, 0.707, 0.707]^T$. The equations of motion are periodic with a period $T_A = 2\pi$, which corresponds to one half of a revolution of the shaft. The sample increment was chosen to be $\Delta t = 0.12566$, corresponding to 50 samples per half revolution of the shaft. The impulse response was found using time integration, via Matlab's "ode45" function. The response was evaluated over a time window encompassing 512 revolutions of the shaft, which was adequate to allow the impulse response to decay to a small fraction of its initial amplitude. The response was then contaminated with Gaussian white noise whose standard deviation was equal to 5% of the maximum response in each coordinate, to simulate a somewhat more realistic scenario. Also, the states corresponding to the velocities of the masses were not used in the identification process; only the positions x and y were made available.

Figure 3 shows the noise contaminated time response of the two position state variables x and y. An inset is also provided where markers highlight the response points at which the turntable (or shaft) is at either 0 or 180 degrees. As discussed previously, this latter collection of points can be described by an LTI, discrete time system. One should also note that this collection of points aliases the true response frequency.

The discrete Fourier transform of the signals was found and is shown in the upper pane of Figure 4. The response is dominated by two resonance peaks at about 0.8 rad/s. Two other pairs of peaks are also seen at 0.2 and 1.8 rad/s, although the latter are almost completely masked by the noise. Hence, this two-degree of freedom LTP system appears to have six active modes in this impulse response. This phenomenon can be interpreted in light of eq. (9), which states that the time varying mode vectors comprising the STM give the appearance of additional modes.

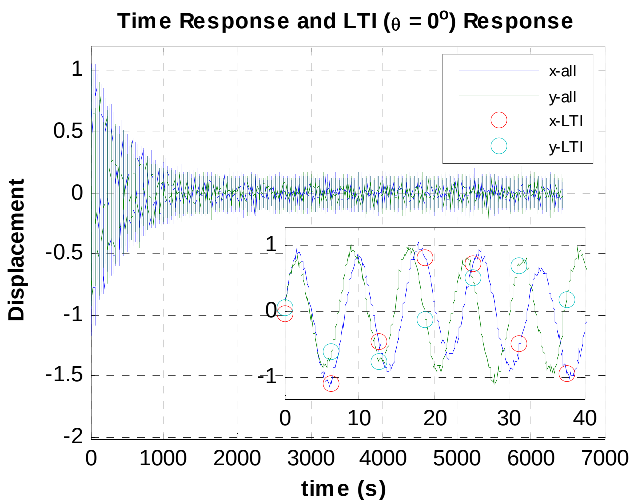
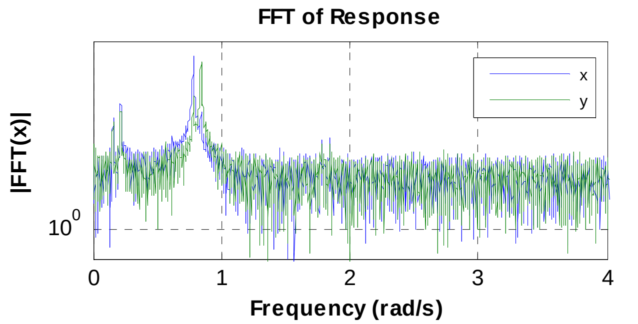


Figure 3: Noise contaminated time response of LTP system. Inset shows the response over the first few cycles. Circles mark the response points for which the shaft is at an angle of zero degrees.



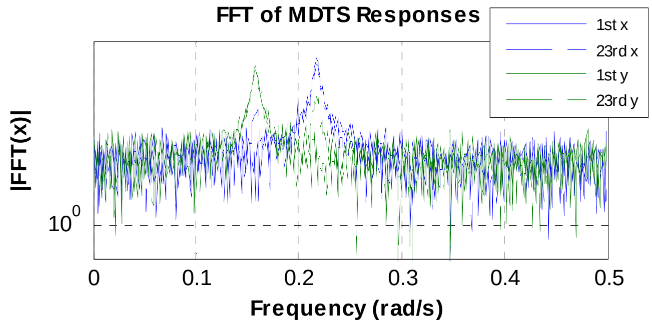


Figure 4: (top) DFT of time response. (bottom) DFT of two MDTS responses.

The response was also decomposed according to the MDTS method, collecting the samples of the response for which the system matrices are the same (i.e. at 0+θ and 180+θ degrees). The response was sampled fifty times per half revolution of the shaft, so this results in 50 sets of time responses in both the x and y directions. (Figure 3 highlighted the first few points of one of these time responses.) The DFTs of two of these time responses are shown in the bottom pane of Figure 4. The 1st and 23rd set of responses are shown corresponding to instants when the shaft was at 0° or 180° (1st x and y) and 79.2° or 259.2° (23rd x and y). Each FFT shows at most two resonant peaks, and the amplitude of the peaks is seen to vary with shaft angle.

Either of the sets of responses in Figure 4 could be used to identify a response model for the system. The MDTS method was chosen because it can be implemented using standard system identification techniques for LTI systems. To perform identification on the full response (in the upper pane of Figure 4) is not as straightforward since it would require either: 1.) customizing an identification routine to enforce a constraint that the identified eigenvalues for each mode have the same real parts and imaginary parts that differ by integer multiples of ω_A, as proscribed by eq. (9) or 2.) post processing the results obtained using a standard system ID routine to enforce these constraints.

The set of 100 MDTS response FFTs were processed using the Algorithm of Mode Isolation [15-18], which considered all sets of responses simultaneously, and automatically identified both of the modes of the system, assuring that the eigenvalues identified were appropriate to the total set of measurements.

The respective residues for each response point-shaft angle combination were also identified by AMI, and the algorithm verified that only two modes were present in the response by observing that the response was reduced to noise after removing these modal contributions from the data. The identified eigenvalues were $\lambda_{aliased}$ = -0.00202+ i*0.158 and -0.00202 + i*0.218.

At this point one must un-alias these eigenvalues in order to proceed. This is done by returning to the top pane in Figure 4 and noting that the two dominant peaks occur at 0.78 and 0.84 rad/s. Assuming that these peaks correspond to m = 0, or that the system's time varying terms are smaller than its time-invariant terms, we deduce that the un-aliased eigenvalues are $\lambda_r = (\lambda_{aliased})^* + 1*\omega_A$. The associated residue vectors for each of these modes can now be transformed to a common starting time of zero as described in the Appendix. The real and imaginary parts of the resulting residues as a function of shaft angle are shown in Figure 5. Both the real and imaginary parts in most of the directions vary significantly with shaft angle in a sinusoidal manner, although the estimated residues are somewhat jagged, presumably due to the effect of the noise added to the responses.

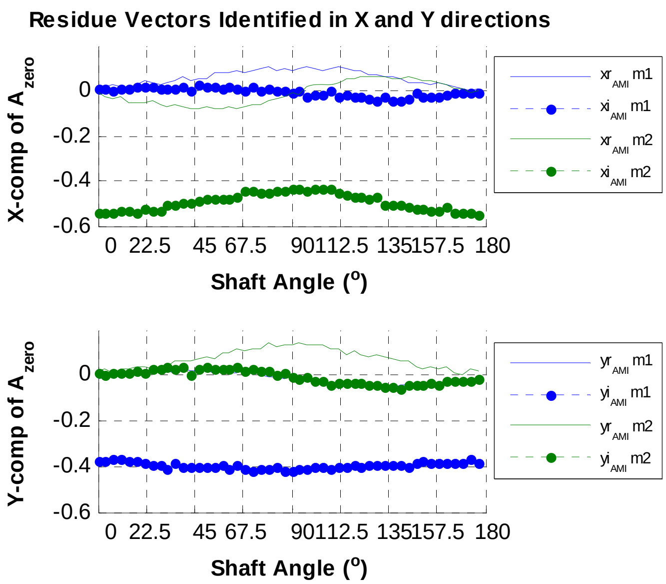


Figure 5: Residue vectors identified by AMI algorithm for LTP system using MDTS processing.

Since two independent states have been measured, and the system has only two modes, the response model identified can be used to generate the state transition matrix and the state matrix for the LTP system using the methods developed in Section 2.3. The following subsection applies these methods to the identified response model and evaluates the results.

3.2. State Transition and System Matrices

The State Transition Matrix (STM) and the time varying system matrix were derived from a Fourier series (FSE) model for the response in a Section 2.3. An FSE model was derived from the MTDS model identified in Section 3.1 in order to apply the relations in Section 2.3. Fortunately, this is easily done by simply finding a Fourier Series expansion of the MTDS residues and then retaining only the significant coefficients. This was implemented using a DFT algorithm, and three terms found to be significant for each residue (N_R=1).

The resulting coefficients were then used to reconstruct the state transition matrix using the following procedure. The Fourier Series model for the residue matrices provided the $\{B\}_{r,m}$ coefficients required in eq. (19) to define the Floquet mode shapes $\{\psi(t)\}$. These were evaluated at zero and inserted into eq. (18) to find the modal participation factors $\{L_r(t_0)\}$. The mode shapes and modal participation factors then define the state transition matrix via eq. (8). This representation is easily differentiated, resulting in a new set of coefficients $\{B_{diff}\}_{r,m}$ defining the Floquet mode shapes of the derivative of the state transition matrix. These two models were evaluated at fifty instants over the fundamental period of the LTP system (i.e. in the time interval $[0,2\pi)$), and eq. (21) was used to solve for the system matrix $A(t)$.

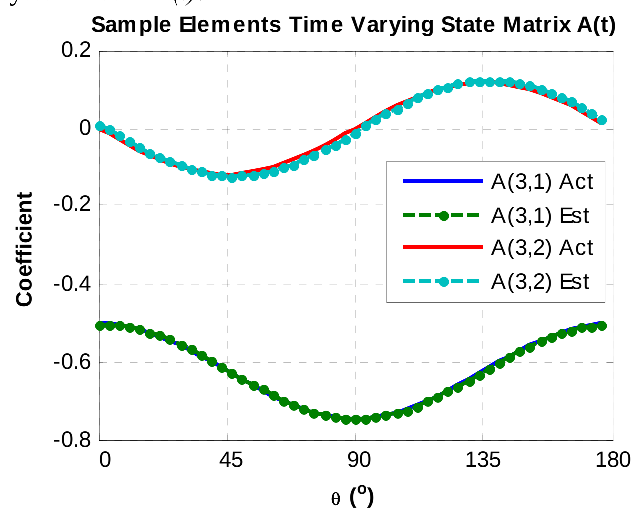


Figure 6: Sample coefficients of the time varying system matrix A(t). (solid) Actual coefficient as a function of shaft angle, (dash-dot) Coefficient estimated from noise contaminated response data.

Figure 6 shows two coefficients of the state matrix estimated using this procedure, compared with the actual coefficients from the analytical model used to generate the response. The A(3,1) and A(3,2) coefficients correspond to the (1,1) and (1,2) elements of the stiffness matrix for this system. The horizontal axis in the figure shows the shaft angle, which is related to time by $\theta = \Omega t$. Observe that both of the coefficients were very accurately estimated using the proposed procedure, even though the response data used was contaminated with a significant level of noise.

4. CONCLUTIONS

A method was presented that identifies the time varying state matrix and state transition matrix of a linear time-periodic (LTP) system from free response data. The method was then demonstrated using synthesized, noise contaminated response data from a simple LTP system. A response model was first fit to the response data using a standard system identification routine for linear time-invariant (LTI) systems and the methods presented in [14]. The state transition matrix $\Phi(t,t_0)$ and accompanying time-periodic system matrix A(t) were then derived for the chosen states. The method was shown to provide highly accurate estimates of the coefficients of the system matrix, even though the responses used to find them were contaminated with substantial noise. Furthermore, the most difficult aspect of the procedure, identifying the eigenvalues of the LTP system and its model order from the response data, was easily performed using a standard system identification routine for LTI systems. These methods could be applied to detect asymmetry (e.g. due to a crack) in the shaft of a rotor-bearing system, to experimentally derive models for complex rotating machines such as wind turbines or helicopters, and to validate analysis models of LTP systems.

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ANNEX A

1. METHODS FOR IDENTIFYING THE PARAMETERS OF AN LTP SYSTEM

In this section we shall present two methods by which the free response of a periodically time varying system can be represented exactly by that of an LTI system. The first method consists of discretizing the PTV system over its fundamental period, resulting into a collection of LTI systems of the same order as the PTV system, corresponding to different portions of the fundamental period. The second method expands the modal matrix of the Floquet representation in a Fourier series, resulting in a single, although possibly high order, LTI representation for the PTV system. In either case, the parameters of the LTI system (or collection of systems) can be identified from the transformed response using standard system identification techniques. The parameters identified for the LTI system are then easily related to those of the PTV system. The relative merits of these two methods will be discussed throughout the remainder of the paper.

1.1. Method #1: Multiple Discrete-Time Systems (MDTS)

From the state transition matrix representation in eq. (4), one can construct a discrete time system whose response matches that of the LPTV system at the instants $t_0 + nT_A$ where n is an integer. First define $x(n) = x(nT_A+t_0)$ where $n = 0, 1, 2, \ldots$ Then one can see that

$$x(n+1) = \Phi(t_0 + T_A, t_0)x(n) = A_D(t_0)x(n) \tag{22}$$

where the matrix $A_D(t_0)$ is constant for a given initial time t_0. This shows that the samples at instants separated by integer multiples of T_A are related by a linear time invariant system whose parameters depend only on the initial time t_0. As a result, one can identify the parameters of the matrix $A_D(t_0)$ using standard methods for LTI systems, so long as the samples are taken at instants separated by integer multiples of T_A.

This method requires that one sample synchronous with T_A. However, it might be desirable to sample the response at a higher or lower rate. This can be achieved by setting the time increment $\Delta t = T_A/P$ or $\Delta t = P*T_A$, where P is an integer. Setting $\Delta t = P*T_A$ corresponds to sampling once every P periods of the system. The system matrix identified from such a response would then be $(A_D)^P$.

The more interesting situation occurs when one samples at a faster rate (such that $\Delta t = T_A/P$). In this case one can learn something about the time varying nature of $A(t)$. In this case, the response is separated into the following collection of responses.

$$\begin{aligned}
y_0 &= [t_0, t_0 + T_A, t_0 + 2T_A, \cdots, t_0 + N_c T_A] \\
y_1 &= [t_1, t_1 + T_A, t_1 + 2T_A, \cdots, t_1 + N_c T_A] \\
&\cdots \\
y_{P-1} &= [t_{P-1}, t_{P-1} + T_A, t_{P-1} + 2T_A, \cdots, t_{P-1} + N_c T_A]
\end{aligned} \tag{23}$$

Each response y_K can be parameterized by a matrix $(A_D)_k$, pertaining to a different initial time t_k where $t_k = k*T_A/P$ for k = 0,1,…P-1. The system parameters can then be determined by applying a standard, LTI modal parameter identification routine to each response y_k independently. In the time domain, for example, one might use the Least Squares Complex Exponential method or a Subspace method [19, 20]. If the responses are transferred to the frequency domain, the Least Squares Complex Frequency Domain Algorithm (LSCF) [21] or the Algorithm of Mode Isolation (AMI) [15, 22-24] could be used.

Some of the aforementioned algorithms estimate the system matrix $(A_D)_k$ directly whereas others identify the modal parameters that comprise it. The modal parameters are related to the system matrix as follows

$$A_D(t_k) = (A_D)_k = M_k \Lambda_k M_k^{-1} \tag{24}$$

where M_k is the modal matrix whose columns contain the state space mode vectors and Λ_k is a diagonal matrix of eigenvalues.

Comparing equation (24) with equation (7) and recalling that A_D is a state transition matrix, one can see that when the Floquet state transition matrix in eq. (7) is diagonalizable, the eigenvalues of each matrix $(A_D)_k$ should be equal (i.e. $\Lambda_k = \Lambda = exp(\Lambda_R T_A)$ for all k) and their respective mode vectors should be the periodic Floquet mode vectors -. In such a case, one can readily relate the modal parameters of the systems $(A_D)_k$ to the Floquet-Lyapunov representation. This was found to be the case for the PTV system presented herein using a variety of different parameter sets.

When the eigenvalues Λ_k of $(A_D)_k$ are not a function of k, the modal parameters of the systems $(A_D)_k$ can be found in a single pass using a "common denominator" parameter identification algorithm. The set of responses $\{y_k\}$ are processed as if they resulted from a single SIMO experiment with N_o*P outputs. This results in a global estimate of the eigenvalues, and an estimate of the mode shapes for each system $(A_D)_k$. For some systems this treatment might entail processing data from a large number of outputs simultaneously, in which case the authors recommend using either the AMI or the LSCF algorithms.

At this juncture it is important to note that the imaginary parts of the eigenvalues of the matrices $(A_D)_k$, even when constant with k, can differ from the Floquet eigenvalues by an integer multiple of the fundamental frequency of the parameters of the LTP system $\omega_A=2\pi/T_A$. This aliasing phenomenon is a well known feature of Floquet-Lyapunov theory [2]. Moreover, it is reasonable considering that the matrices $(A_D)_k$ are only related to the system response at the instants $t_k + nT_A$. However, if one considers the response at all time instants (i.e. for $[t_0, t_1, \ldots, t_P, t_{P+1}]$), one can usually determine the integer multiple that relates the aliased eigenvalues to the true Floquet eigenvalues, as will be elaborated in the following sections.

It is important to note that although that the residues identified from the collection of LTI systems used in Method #1 are proportional to the mode vectors when taken individually, the constant of proportionality for each residue is different. They must be rescaled so that the constant of proportionality is not a function of shaft angle. To do this, we appeal to the summation form of the Floquet representation of the response in eq. (30). Applying the initial conditions to eq. (30), the impulse response in terms of the modal parameters follows.

$$x(t) = \sum_{r=1}^{2N} \left(A_R\right)_r \exp\left(\lambda_r (t - t_0)\right)\left\{x(t_0)\right\} \tag{25}$$

The responses at time instants separated by T_A, which are processed by using Method #1, follow.

$$\begin{aligned} x(nT_A + t_0) &= \sum_{r=1}^{2N} \left(A_R(t_0)\right)_r \mathrm{e}^{\left(\lambda_r (nT_A - t_0)\right)} \left\{x(t_0)\right\} \\ x(nT_A + t_1) &= \sum_{r=1}^{2N} \left(A_R(t_1)\right)_r \mathrm{e}^{\left(\lambda_r (nT_A - t_1)\right)} \mathrm{e}^{\left(\lambda_r (t_1 - t_0)\right)} \left\{x(t_0)\right\} \\ x(nT_A + t_2) &= \sum_{r=1}^{2N} \left(A_R(t_2)\right)_r \mathrm{e}^{\left(\lambda_r (nT_A - t_2)\right)} \mathrm{e}^{\left(\lambda_r (t_2 - t_0)\right)} \left\{x(t_0)\right\} \\ &\cdots \end{aligned} \tag{26}$$

The strategy in Method #1 is to process these responses globally, as if they resulted from a single LTI system. As a result, the identification routine finds the following representation for the impulse response:

$$\begin{aligned} x(nT_A + t_0) &= \sum_{r=1}^{2N} \left(A_{Ident}\right)_r \exp\left(\lambda_r (nT_A - t_0)\right)\left\{x(t_0)\right\} \\ x(nT_A + t_1) &= \sum_{r=1}^{2N} \left(A_{Ident}\right)_r \exp\left(\lambda_r (nT_A - t_1)\right)\left\{x(t_0)\right\} \\ x(nT_A + t_2) &= \sum_{r=1}^{2N} \left(A_{Ident}\right)_r \exp\left(\lambda_r (nT_A - t_2)\right)\left\{x(t_0)\right\} \\ &\cdots \end{aligned} \tag{27}$$

Comparing eq. (26) and (27) reveals that the residues identified via Method #1 for various initial times t_i are related to the true residues as follows.

$$\left(A_{Ident}\right)_{r,i} = \left(A_R(t_i)\right)_r \exp\left(\lambda_r (t_i - t_0)\right) \tag{28}$$

If the Floquet eigenvalues are known, it is trivial to solve for the Floquet residues, which are denoted A_{Zero} because they (for all i) relate to the same initial condition at t_0.

$$\left(A_{zero}\right)_{r,i} = \left(A_{Ident}\right)_{r,i} \exp\left(-\lambda_r (t_i - t_0)\right) \tag{29}$$

1.2. Method #2: Fourier Series Expansion (FSE)

The Floquet representation of the response in eq. (7) can be expressed in summation form as follows

$$\begin{aligned} \Phi(t, t_0) &= \sum_{r=1}^{2N} \left(A_R\right)_r \exp\left(\lambda_r (t - t_0)\right) \\ \left(A_R\right)_r &= \left\{\psi_r(t)\right\}\left\langle L_r(t_0)\right\rangle \end{aligned} \tag{30}$$

where N denotes the number of modes, $\psi_r(t)$ are the mode vectors or the columns of $[\Psi(t)]$, and $L_r(t)$ are the modal participation factors, or the rows of $[\Psi(t)]^{-1}$. The residue matrices $(A_R)_r$ are the product of a periodic column vector and a periodic row vector, and hence are themselves periodic. Because they are periodic, they can be readily expanded in a Fourier series. Here we shall assume that they can be adequately represented using a fixed number $(2*N_R + 1)$ of terms so that the response may be written as

$$\Phi(t, t_0) = \sum_{r=1}^{2N} \left(\sum_{m=-N_R}^{N_R} \left[B\right]_{r,m} e^{(im\omega_A (t - t_0))} \right) e^{\left(\lambda_r (t - t_0)\right)} \tag{31}$$

where $[B]_{r,m}$ is the mth complex Fourier coefficient of the rth residue matrix and $\omega_A = 2\pi/T_A$. Factoring out the summations reveals the nature of the response.

$$\Phi(t, t_0) = \sum_{r=1}^{2N} \sum_{m=-N_R}^{N_R} \left[B\right]_{r,m} e^{\left((\lambda_r + im\omega_A)(t - t_0)\right)} \tag{32}$$

This can be thought of as the impulse response of an LTI system with $2*N*(2*N_R + 1)$ eigenvalues $\lambda_r + i*m*\omega_A$. The amplitude of each mode's response is determined by the magnitude of the Fourier coefficient of its residue matrix. The response and hence the state transition matrix must be real, so one can see that every complex eigenvalue must be accompanied by it's complex conjugate and that the Fourier coefficients of the residue matrices must also be real or part of a complex conjugate pair.

The response in eq. (32) is indistinguishable from the response of a state-space LTI system having eigenvalues $\lambda_r + i*m*\omega_A$. Hence, one can identify the parameters in eq. (32) from the time response or FFT of the time response directly using any standard parameter identification algorithm for LTI systems. Equation (32) can then be used to interpret the result and/or reconstruct the Floquet representation.

2. EQUATIONS OF MOTION FOR SIMPLE LTP SYSTEM (JEFFCOTT ROTOR)

The equations of motion follow where the state vector contains the response in the x and y directions in the fixed reference frame.

$$\begin{aligned}
&[M]\begin{Bmatrix}\ddot{X}\\ \ddot{Y}\end{Bmatrix}+[C]\begin{Bmatrix}\dot{X}\\ \dot{Y}\end{Bmatrix}+[K_{PTV}]\begin{Bmatrix}X\\ Y\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}\\
&[M]=\begin{bmatrix}m & \\ & m\end{bmatrix},\ [C]=c_f\,[K_F]\\
&[K_{PTV}]=[R][K_R][R]^T-[R][K_R][R]^T\times\\
&\left([R][K_R][R]^T+[K_F]\right)^{-1}[R][K_R][R]^T\\
&[K_R]=\begin{bmatrix}k_{Rx} & \\ & k_{Ry}\end{bmatrix},\ [K_F]=\begin{bmatrix}k_{Fx} & \\ & k_{Fy}\end{bmatrix}\\
&[R]=\begin{bmatrix}\cos(\Omega t) & \sin(\Omega t)\\ -\sin(\Omega t) & \cos(\Omega t)\end{bmatrix}
\end{aligned} \tag{33}$$

Note that the damping matrix is proportional to the turntable stiffness matrix via the factor c_f.