Nonlinear Normal Modes of Geometrically Nonlinear Finite Element Models about Thermal Equilibrium States

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Abstract

Advanced high-speed aircraft will operate in harsh environments that contain both high external fluctuating pressures and extreme thermal loads. The thermal load changes along the trajectory of the vehicle's mission, but the dynamics of the thermal response typically evolve at a much slower rate than the structural dynamics, so often the dynamic response can simply be studied at each temperature state of interest. Linear modal analysis can be conducted about the thermal equilibrium, revealing the frequencies and mode shapes for each state. However, those modal parameters are only valid for small deformations near the equilibrium state. Nonlinear normal modes provide insight into how the nonlinear dynamics of the structure will evolve with amplitude. This work investigates the nonlinear normal modes of geometrically nonlinear structures about non-zero thermal equilibrium states. The nonlinear normal modes are computed using the Multi-Harmonic Balance (MHB) method, applied directly to finite element models using a new Matlab-based open-source finite element platform called OsFern. The method is demonstrated on a flat beam with fixed boundary conditions in this abstract and other structures will be discussed in the presentation.
Keywords: Finite Element Analysis, Nonlinear Dynamics, Nonlinear Normal Modes, Multi-Harmonic Balance, Thermal Stresses

1 Introduction

Thin shell structures such as panels on high-speed aircraft are subject to both high temperature environments and large acoustic excitation [1]. To design these thin structural components, thermal-elastic analysis of the dynamic response to such loading conditions is required. A common approach in engineering design and analysis is to investigate the linear dynamics about a thermal equilibrium state using the finite element (FE) method. This procedure typically involves computing the static response, linear or nonlinear, to an applied temperature field followed by a modal analysis about the thermal equilibrium state. If external forces are small enough that the response remains linear about the equilibrium state then conventional linear modal analysis may suffice. However, if the external excitation is large enough to cause nonlinear oscillations about the equilibrium state then more advanced analysis is required.

If nonlinear dynamic analysis about the thermal equilibrium is required, the typical approach is to directly integrate the finite element model in time. This can be a costly approach, and it typically gives little insight into the underlying dynamics of the system. Jarman, Van Damme, and Allen [2] demonstrated that the dynamics of geometrically nonlinear structures about a thermal equilibrium could be investigated by looking at the structure's nonlinear normal modes (NNMs). In that work the NNMs were computed from reduced order models created from the parent FE model. The NNMs of a structure provide information into each mode's amplitude-frequency dependence, which characterize its nonlinear dynamic behavior. This work differs from previous analysis [2] in that it investigates the nonlinear dynamics using nonlinear normal modes computed directly from the full FE model about the thermal equilibrium state. The NNMs are computed using the multi-harmonic balance (MHB) method applied directly to the FE model, which provides an efficient approach to compute the NNMs about the thermal equilibrium state.

2 Theory

The geometrically nonlinear thermal-elastic FE model equation of motion for an n degree of freedom (DOF) system can be written as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \left(\mathbf{K} + \mathbf{K}_{\sigma}\left(\mathbf{x}, \Delta T\right)\right)\mathbf{x} + \mathbf{f}_{nl}\left(\mathbf{x}\right) = \mathbf{f}_{\mathbf{ext}} + \mathbf{f}_{\Delta T}\left(\mathbf{x}, \Delta T\right) \quad (1)$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are the mass, damping and linear stiffness matrices respectively; $\mathbf{f}_{nl}(x)$ is the nonlinear restoring force vector and $\mathbf{f}_{ext}$ is the $\mathrm{n} \times 1$ external force vector. The thermal portions of the model are contained within the term $\mathbf{K}_{\sigma}$, which is the stress-stiffening matrix caused by the thermal stresses and $\mathbf{f}_{\Delta T}$, which are the induced thermal loads. In this work the stress stiffness matrix and the thermal force are defined as functions of both temperature and the current deformation, i.e. $\mathbf{K}_{\sigma} = \mathbf{K}_{\sigma}(\mathbf{x}, \Delta T)$ and $\mathbf{f}_{T} = \mathbf{f}_{T}(\mathbf{x}, \Delta T)$ respectively, leading to a nonlinear, coupled problem. These equations of motion can be directly integrated in many commercial FE codes but can be computationally expensive; we will thus compute the NNMs in this work.

A nonlinear normal mode, as defined by Kerschen et al. [3] and used here is a *non-necessarily synchronous periodic motion of the conservative system* that provides information into the amplitude-frequency dependence of a nonlinear system. The NNMs for the undamped version of Eq. (1) with no external forces are periodic solutions of the following equation,

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{K} + \mathbf{K}_{\sigma}(\mathbf{x}, \Delta T))\,\mathbf{x} + \mathbf{f}_{nl}(\mathbf{x}) = \mathbf{f}_{\Delta T}(\mathbf{x}, \Delta T) \tag{2}$$

that originates from the linear normal modes computed about the thermal equilibrium.

The MHB method is used in this work to compute periodic solutions of the undamped, unforced nonlinear system defined in Eq. (2) in which thermal stress stiffening and forces are included. The periodic motion of the original equation in the time domain can be solved efficiently by representing the response as a Fourier series, inserting that series into the equation of motion, and solving the resulting system of nonlinear algebraic equations. The setup of the MHB method will follow similarly to that found in the anlaysis by Detroux et al. [4]. Assuming that the response, $\mathbf{x}(t)$, and the sum of the geometric nonlinear force, thermal force and force taken by the stress stiffness matrix, $\mathbf{f} = \mathbf{f}_{nl}(\mathbf{x}) - \mathbf{f}_{\Delta T}(\mathbf{x}, \Delta T) + \mathbf{K}_{\sigma}(\mathbf{x}, \Delta T)\,\mathbf{x}$, are periodic, they can be represented as a sum of Fourier coefficients truncated to N_H harmonics written as

$$\mathbf{x}(t) = \mathbf{z}_0 + \sum_{h=1}^{N_H} \left(\mathbf{z}_s^{(h)} sin\,(h\omega t) + \mathbf{z}_c^{(h)} cos\,(h\omega t) \right) \tag{3}$$

$$\mathbf{f}(t) = \mathbf{F}_0 + \sum_{h=1}^{N_H} \left(\mathbf{F}_s^{(h)} sin\,(h\omega t) + \mathbf{F}_c^{(h)} cos\,(h\omega t) \right) \tag{4}$$

where ω is the frequency of oscillation, and h is the order of the harmonic After inserting those terms into the equation of motion, performing algebraic manipulations and projecting the equation onto an orthogonal trigonometric basis, the result is a set of nonlinear algebraic equations that is a function of frequency and the Fourier coefficients, $\mathbf{z}$. For details on the derivation the reader is referred to Detroux et al. [4]. The final form of equations is

$$\mathbf{h}(\mathbf{z}, \omega) = \mathbf{A}(\omega, \Delta T)\mathbf{z} + \mathbf{F}(\mathbf{z}, \Delta T) = \mathbf{0}, \tag{5}$$

in which the linear term, $\mathbf{A}(\omega)$, is

$$\mathbf{A}(\omega) = \nabla^2(\omega) \otimes \mathbf{M} + \mathbf{I}_{(2N_H+1)} \otimes \mathbf{K} \tag{6}$$

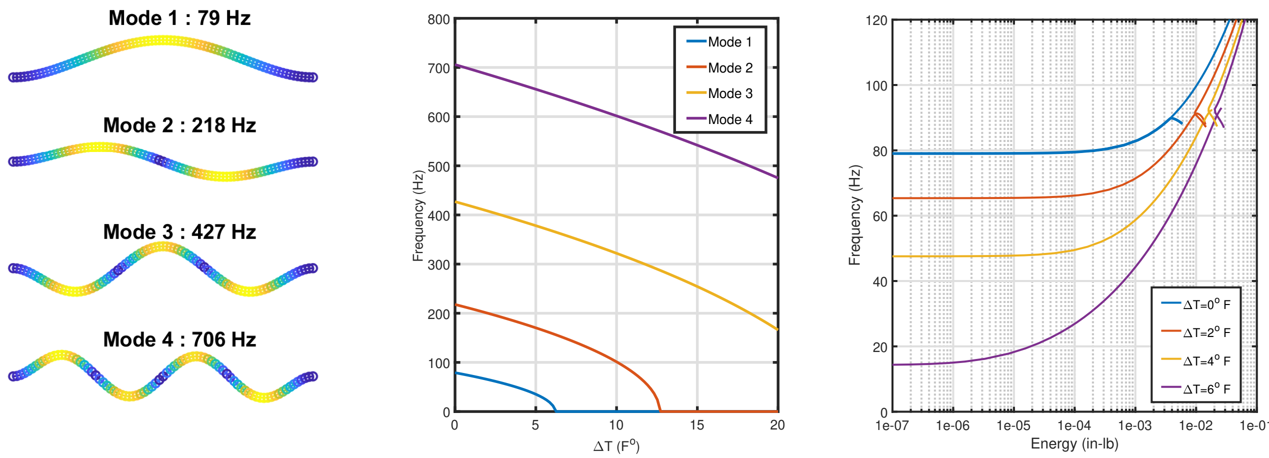
and the external force and nonlinearities are captured within $\mathbf{F}(\mathbf{z}, \Delta T)$. The nonlinear term is obtained by transforming the time-domain force vector to the frequency domain by representing the Fourier transformation as a linear operator [4]. The MHB method is coupled with a pseudo arc-length continuation routine to compute the NNM branches that originate from the linear normal modes about the thermal equilibrium state.

3 Numerical Example

The numerical example used here is a steel flat clamped-clamped beam that has been investigated in many works [1,5]. The beam has a length, of 228.6mm (9 in), a width of 12.7 mm (0.5 in), a thickness of 0.7874 mm (0.031 in), a modulus of elasticity of E = 2.07× 10^5 N/mm^2 (2.97 × 10^7 lb/in^2), a density of ρ = 7.8 × 10^{-6} kg/mm^3 (7.36 × 10^{-4} $lb \cdot s^2/in^4$), Poisson's ratio ν = 0.29, and coefficient of thermal expansion of 6.3 × 10^{-6} $in/in/^oF$. The beam was modeled in an in-house finite element code with 80 two-node beam elements and restrained to in-plane motion resulting in a total of 242 DOF.

The first four linear normal modes are presented in subplot (a) of Figure 1. A uniform temperature field was applied to the beam that creates a stress-stiffening matrix and internal force vector. The linear normal modes are dependent on the thermal stress stiffness matrix and thus the temperature of the structure. For a perfectly flat structure, the increase in temperature creates a softening effect that lowers the frequency of the bending modes. This is demonstrated in subplot (b) of Figure 1 which displays the first four linear frequencies of the model as a function of temperature. For each mode the frequencies decrease with an increase in temperature. Eventually, as the temperature field increases to above 6 degrees F, the first mode becomes complex which occurs because the temperature has caused the flat beam to buckle. This work only examines the NNMs prior to the buckled state.

The change in the stress stiffness matrix also has influence on the nonlinear dynamics of the system by influencing the ratio of membrane stretching to bending stiffness. The nonlinear normal mode originating from the first linear normal mode of the flat beam is presented in Figure 1c. The NNM was computed using five harmonics retained in the MHB method. The NNM is represented as a frequency energy plot (FEP). As the temperature applied increases, the NNM shows more nonlinearity for a given energy. We presume that this happens because the stress stiffness matrix reduces the bending stiffness of the beam and hence the ratio of membrane to bending stiffness increases. An interesting thing to note is that, for each temperature state, the 1:5 internal resonance with the third mode always occurs along the backbone near 90 Hz.



(a) (b) (c)

Figure 1: Flat beam numerical example: (a) Linear normal modes about unstressed state, (b) linear frequencies versus temperature, and (c) nonlinear normal mode originating from the first linear normal mode at various temperature states.

4 Conclusion

This work demonstrated the ability to compute NNMs of geometrically nonlinear structures about thermal equilibrium states using the multi-harmonic balance method. The method was applied to a numerical example of a steel flat beam with fixed boundary conditions. A uniform temperature was applied; as it increases, the temperature causes a decrease in frequency until the beam buckles. NNMs were computed over a wide range of amplitude and at a few temperatures, all of which were low enough to keep the beam from statically buckling. The nonlinear dynamics indicated an increase in the hardening of the NNM with increased temperature. Further results for this beam and for other structures will be presented in the conference.

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