Combining Experimental and Analytical Substructures with Multiple Connections

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NOMENCLATURE

A^+	Pseudo inverse of Matrix A
CMS	Component Mode Synthesis
DOF	Degree(s) of Freedom
FE	Finite Element
FRF	Frequency Response Function
HA	FRF Matrix for Subsystem A
HB	FRF Matrix for Subsystem B
HT	FRF Matrix for Total (combined) System
MCFS	Modal Constraint for Fixture and Subsystem
c	Subscript for Connection DOF
i	Subscript for Force Input DOF
p	Subscript for Physical Connection DOF
r	Subscript for Response DOF

ABSTRACT

Combining experimental and analytical substructures at multiple points is complicated by the traditional requirement to measure every translation and rotation at the connection points on the experimental substructure. Typically a separate fixture is required at every connection point to estimate the translations and rotations as well as apply forces and moments. This work presents a method whereby a single, flexible fixture can be utilized for this purpose. The connection responses are represented with modal degrees of freedom of the fixture. For the system considered here, the approach allows reduction of the degrees of freedom at 8 connections from 48 to 18. An analytical copy of the fixture is also connected to a second structure, which is modeled analytically. The two fixtures are then constrained to have the same motion, and the mass and stiffness of the two fixtures is subtracted to predict the total system response. A methodology to reduce the measurement degrees of freedom on the fixture from 48 translations and rotations to a smaller number of measurement translations is presented. The fixture design is intended to assure that the mode shapes of the fixture satisfactorily span the space of the

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connection motion. Encouraging results for real hardware were realized using both admittance modeling and component mode synthesis. The robustness of the modal connection result is contrasted with connection of the substructures through physical degrees of freedom without a fixture.

INTRODUCTION AND MOTIVATION

Structural dynamic models are utilized for developing designs and qualification in many applications. Full system models are often built up from several subsystems. In certain applications, the best use of resources may be to utilize a finite element (FE) model for one subsystem and an experimentally derived model for another subsystem. Various substructuring methods exist that could theoretically be utilized to join these models, but up to now practical limitations, mostly on the experimental substructures, have made it difficult to implement hybrid analytical/experimental substructuring, except in a few applications. Usually, hybrid models are restricted to cases where there is a single assumed connection point between the experimental and analytical system. The obstacles to large scale utilization of experimental substructures have been:

1. Measuring the three translations, three rotations, three forces and three moments at every physical connection point;
2. Representing all the necessary flexibility that will be exercised when the two substructures are attached while also keeping the bandwidth of the experiments within practical limits;
3. Making these measurements for multiple connection points.

Considering this from a component mode synthesis (CMS) approach, these issues are not insurmountable for a FE substructure, as classic techniques such as Craig-Bampton fixed interface CMS models with constraint modes calculations can be made to satisfy all three obstacles. But experimentally, fixed interfaces are impractical, so most experimental models are derived using softly suspended substructures with almost free boundary conditions. The free interface translation mode shapes can be obtained from a classic modal test, but getting the rotations at the connection locations is not automatic. Also, the free modes almost never provide enough accuracy, so additional residual flexibilities have to be measured for all the connection point degrees of freedom. This is usually accomplished by attaching a rigid fixture to a connection point. If there are enough sensors (at least 6 appropriately placed), the translation and rotation responses at the connection can be obtained from rigid body response of the fixture for the free modes. One could also use finite difference approaches to estimate the rotations [3]. To get the residual flexibility requires that 6 forces (at least) be appropriately applied to rigid fixtures at every connection point. Some researchers have performed this for applications with single connection points. Where multiple connection points have been considered, usually the rotations and moments are just neglected. In general, the effort to put a fixture at every connection point and measure all 6 responses and 6 forces and estimate residual flexibilities from these many measurements has been overwhelming. Some have had some reasonable success using the fixtures to provide mass loaded interfaces to capture some additional flexibility, but usually the number of connection points is not much greater than one as with Martinez, Carne, et al[3].

The approach introduced here is to utilize a single fixture to capture all the motions at multiple connection points. However, the fixture considered is not rigid. Instead, it is meant to approximate the boundary condition applied to the test article in the final structure. All the rigid body modes and several elastic modes of the fixture are used to provide an approximation of the connection motion. In conjunction with this approximation, the connections between substructures are made using the modal degrees of freedom of the fixture. Although it is an approximation, this approach allows us to overcome the three obstacles listed above. Even though one can rarely measure the translations and rotations at the assumed physical connection point, one can make measurements at many other places on a fixture that are robust enough to capture the modal motion of the fixture. Also, the modal dof of the fixture inherently capture the rotations at the connection points. Therefore no rotations have to actually be measured. The key is that the mode shapes of the fixture must span the space of the connection motion when the substructures are connected. The fixture provides the added benefit of exercising the joint. Ideally, the fixture would be designed so that all modes necessary to describe the connected motion would be in the testable frequency band. Although this approach is still subject to the approximations due to modal truncation, it retains the benefit of mass loaded modes, removes the necessity to measure rotations, requires only one fixture and reduces the number of connection dof (and constraints) of the substructure from the number of physical dof down to the number of effective degrees of freedom (active modes) of the fixture.

Mayes & Stasiunas [1] and Allen & Mayes [2] recently presented a study coupling an experimental beam to an analytical beam substructure. The experimental approach required a fixture to estimate the rotations and moments at the connection point in the experiment. They showed that the coupling was more accurate when elastic, as well as rigid body modes of the fixture, were included. Even greater improvements were realized when the coupling was accomplished utilizing modal degrees of freedom of the fixture. After this success, it was realized that the modal connection approach could be utilized with a single fixture on a system that had multiple connection points instead of just a single connection point as on the beam. This work shows the results of applying this method to connect two real substructures at 8 connection points around a flange. One substructure is modeled with FE, and the other is derived from a modal test of the experimental substructure and fixture. In the earlier work, Allen & Mayes [2] named this method Modal Constraint for Fixture and Subsystem (MCFS).

MODAL CONSTRAINT FOR FIXTURE AND SUBSYSTEM (MCFS) APPROACH

To review, the approach is to connect a well designed fixture to the experimental substructure at the points it will actually connect to the FE substructure. Enough translation measurements are made on the fixture to capture the modal motions of the fixture (eliminating the need to measure the rotations and translations at every connection point). The other key to MCFS is that an analytical copy of the fixture is connected to the FE substructure. This allows the modal fixture dof of the FE substructure to be coupled to the modal fixture dof of the experimental substructure. After we have connected the two substructures using the modal constraints, the two fixtures are subtracted by coupling the result to two negative fixtures. In this paper, this is accomplished using either CMS equations or admittance modeling with FRFs to see what issues arise in each method.

SYSTEM UNDER STUDY

In this work two substructures were connected. The analytical FE substructure is a cylinder with a flange on top. The experimental substructure is a plate with a beam mounted in the center of the plate. The flange of the cylinder is connected to the plate through 8 bolts equally spaced. Washers between the flange and plate help ensure that there are eight unique connection points. Actual hardware for both substructures was built and tested, and the full system was also tested for a truth measurement. The figures below show the hardware as well as the fixture that was used in the experimental setup and analytically attached to the FE substructure. The goal was to produce driving point FRFs that were accurate to 2000 Hz with this system that had a first elastic mode near 130 Hz. The fixture was modeled with FE and the experimental substructure was modeled using results from a modal test. In addition, the experimental substructure was modeled with FE to provide virtual test results so the CMS and admittance algorithms could be debugged in advance of utilizing the experimental model. One driving point FRF was produced for an axial input to the end of the beam on the experimental substructure, and the other was a driving point on the side of the cylinder about three inches from its base.

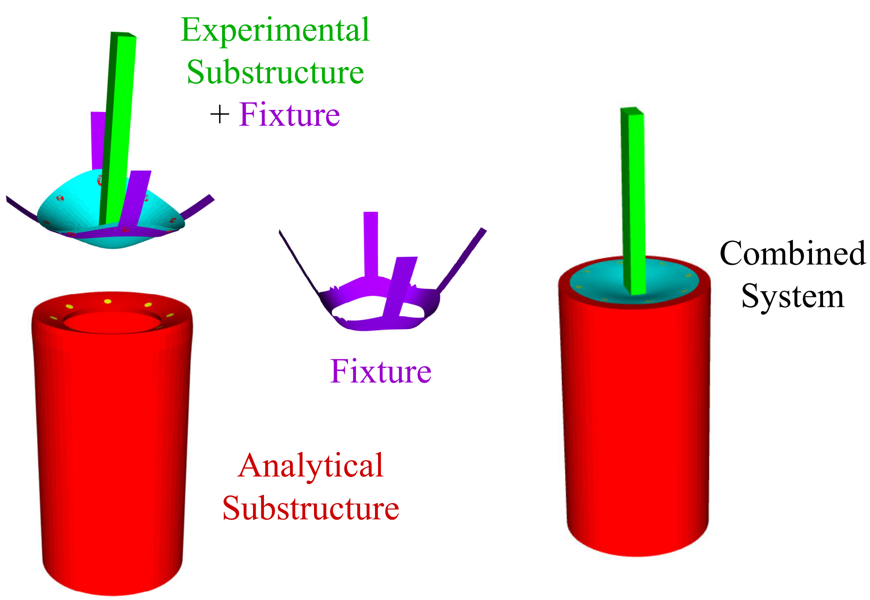

Figure 1 - Representations of two substructures and full system under study

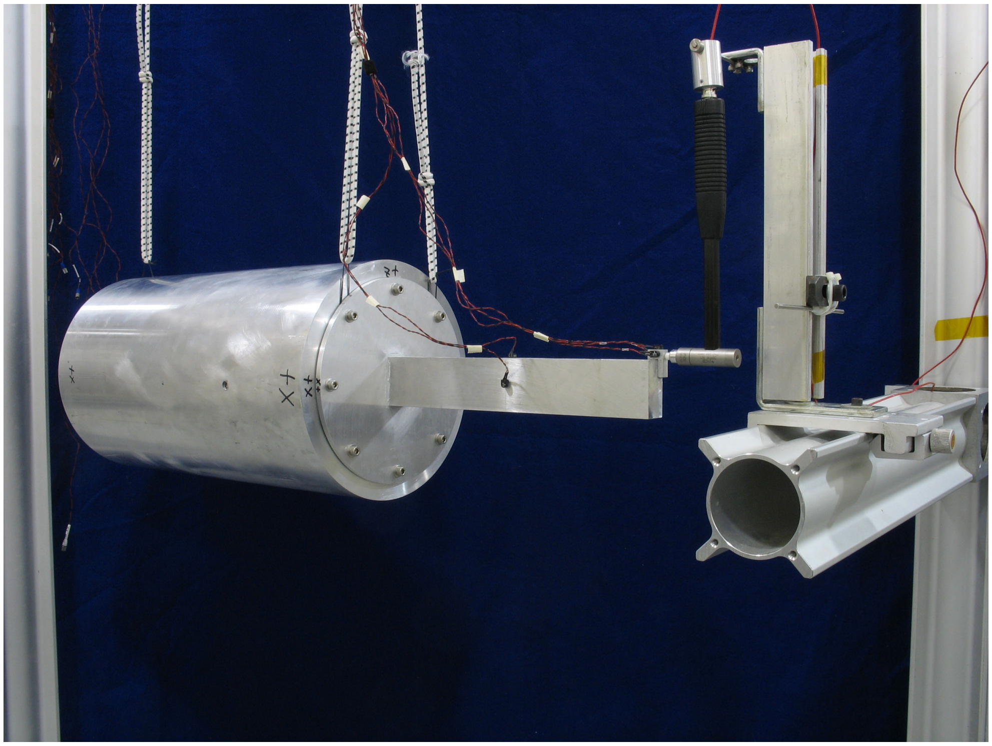

Figure 2 - Experimental setup for full system

RING FIXTURE DESIGN

A basic requirement for this method is that the mode shapes of the fixture must span the space of the motions of the connection points. As more modes of the fixture are utilized, more accurate approximations of the connection motion can be achieved. How many modes are required? In this case, the authors used a heuristic approach. Our engineering sense was that (in addition to the rigid body modes) a dishing mode of the ring was required to represent the axial motion; some bending modes of the ring were needed to capture the bending motion; and the first two ovaling modes might be needed to capture ovaling. A ring with four tabs was designed which bolted to the plate at the eight connection locations (Figure 3). These tabs were designed so that their first bending frequencies would be below 2000 Hz, to make sure they were in the testable frequency band. The tabs ensured that there would be the dishing, bending and ovaling modes in the band below 2000 Hz. A FE model of the ring fixture was generated. Mode shape information of the fixture was generated by the FE model. A quick tap test of the fixture was performed to be sure that the FE model frequencies were reasonable. Heuristically, the fixture was designed to bring modes that exercised the connection down into the testable bandwidth. One tradeoff is that a stiffer fixture brings fewer modes down into the testable bandwidth requiring fewer measurement accelerometers and fewer modes to be extracted, while a softer fixture brings more modes down into the testable bandwidth, requiring more modes to be extracted and more measurement accelerometers. After this work, our sense is that the fixture probably could have been stiffer. Future work will examine stiff and soft fixtures.

SUBSTRUCTURE DEVELOPMENT AND ASSUMPTIONS

The system under study was extremely lightly damped, being fabricated from aluminum with steel bolted connections. The experimental FRFs were synthesized from the experimental modal parameters for the admittance modeling method and the mode shapes and frequencies were utilized directly for the CMS method. Rigid body modes were developed from the measured mass properties. Analysis FRFs were derived from the analysis mode shapes, frequencies and an assumed damping. Damping ratios were assumed at 0.25 percent for all modes of both the analytical and experimental substructure for making predictions, an average value from the results of the experimental substructure modal test. After predictions were made, full system FRFs were measured, and modal extraction estimates of the damping for the full system were from 0.03 to 0.13 percent.

MODAL TEST OF THE EXPERIMENTAL SUBSTRUCTURE

Figure 4 shows the experimental substructure modal test setup. The rigid body modes were obtained from mass properties measurements of the parts. Twenty other elastic modes up to 4000 Hz were extracted from FRFs collected using impact excitation. Many of these modes were different configurations of the fixture tabs moving. Several were bending modes of the beam on the plate. Extracted damping ratios were from 0.05 to 0.8 percent. Four different hammer input locations were utilized to excite the modes. Two were in the axial and soft bending direction at the end of the beam and two were on the plate. Attempts to excite the fixture on the tabs failed due to multiple hits. The multi-reference SMAC algorithm[5] was utilized to extract the modal parameters. Five accelerometers on the beam, four triaxes on the tabs and 8 triaxes mounted on the fixture ring were acquired in addition to two more plate driving point accelerometers. FRFs and the Complex Mode Indicator Function (CMIF) were synthesized from the modal parameters and compared to the experimental data to ensure a high quality set of modal parameters. The CMIF comparison is shown in Figure 5, where the dashed lines are synthesized from the modal parameters and the solid lines are the experimental data.

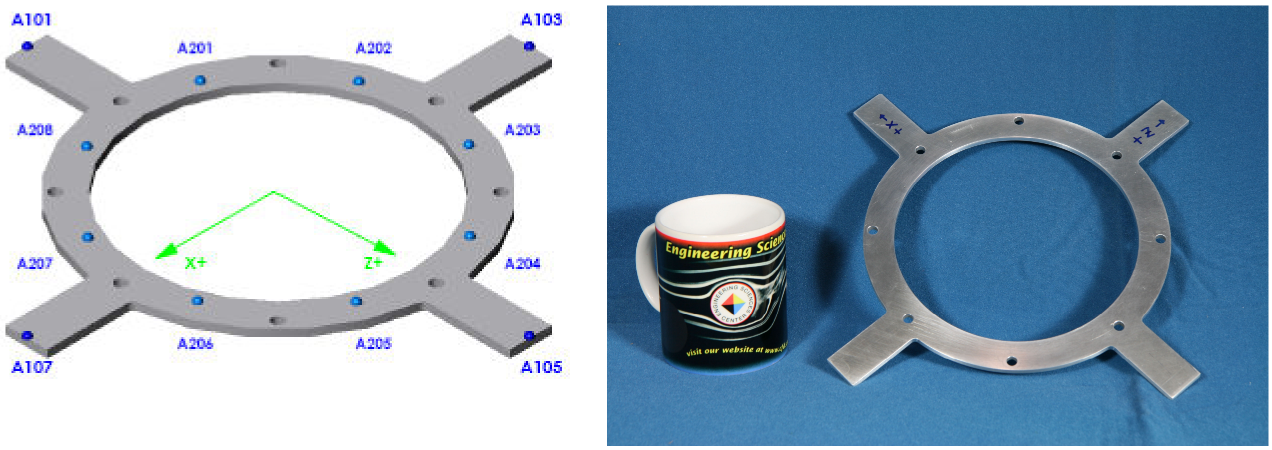

Figure 3 - Fixture and accelerometer locations

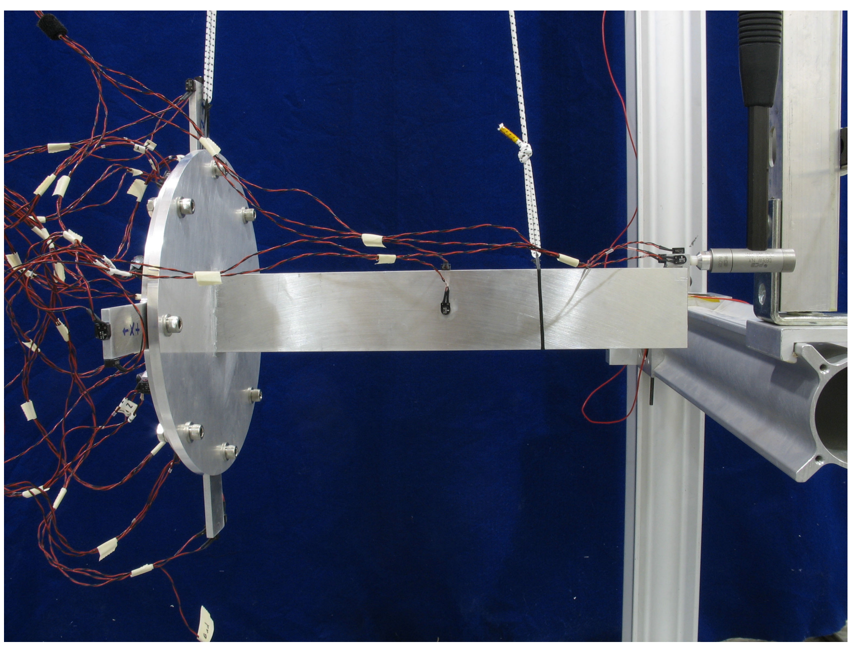

Figure 4 - Modal test setup for experimental substructure with fixture

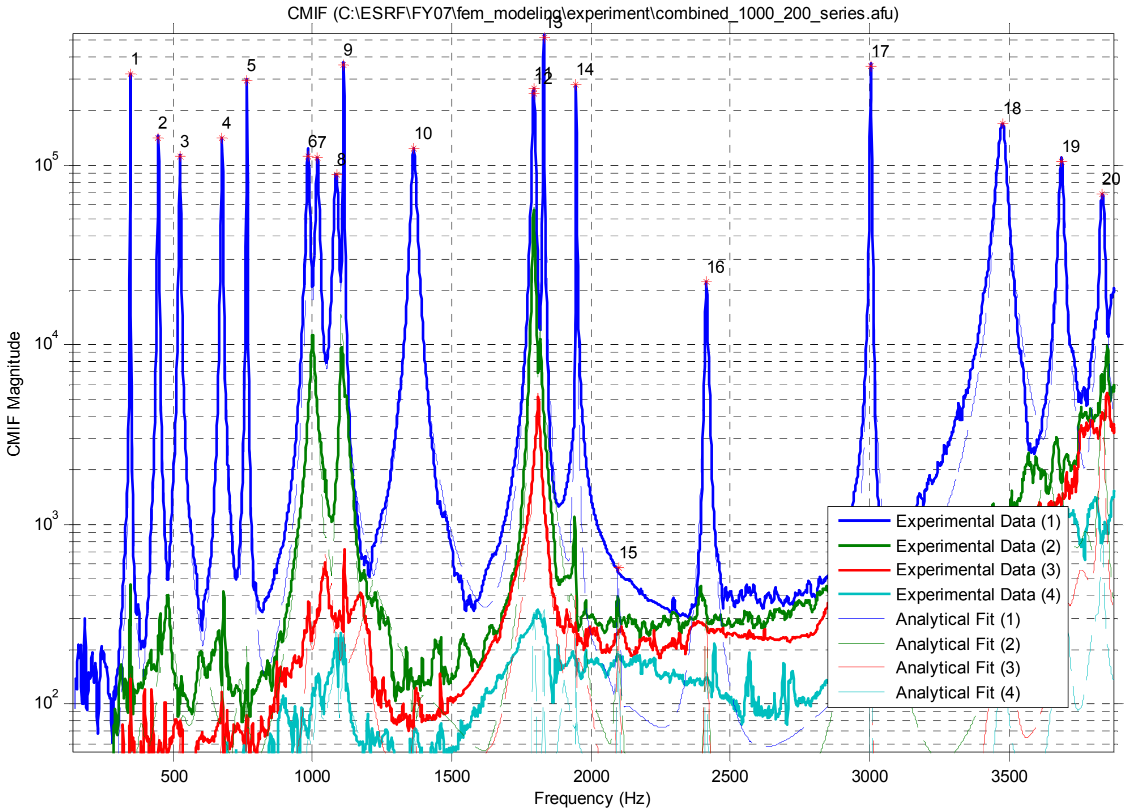

Figure 5 - Analytical Synthesis of Imaginary Part of CMIF Compared to Data

THEORY

The theory for the MCFS approach using Admittance Modeling is presented in Appendix A. The theory for MCFS using CMS was presented by Allen & Mayes [2].

FIXTURE SENSORS

One advantage of this approach is that one does not have to measure all 48 physical connection translation and rotation responses. The connection of the substructures is made with the generalized modal dof of the fixture. A fixture is mounted to the experimental substructure, and an analytical fixture is attached to the FE substructure. Then the fixtures are constrained to have the same motion to make the connection. The requirement for accelerometers is that they adequately capture the fixture motion in the chosen fixture mode shapes. This study was performed analytically with several candidate triaxial sensor locations given by the fixture FE model. The authors chose the first sixteen modes as being adequate to span the connection point response space. The algorithm to choose the sensor locations was to take the mode shapes of all candidate sensor locations for the sixteen modes and throw away one sensor. Then the condition number of the mode shape matrix was calculated. This was repeated throwing away another candidate sensor and retaining the first sensor. After this was done for all candidate sensors, the sensor that increased the condition number of the mode shape matrix the least was thrown away. The process was repeated again to throw away the next sensor and so on until the sensor set was reduced to 36 (12 triaxes). In Figure 6 below, the condition number of the FE mode shape matrix versus number of fixture mode shapes is shown for the 12 triaxes that were used. The optimization was for 16 shapes, and the condition number is relatively low up through 17 mode shapes, but begins to rise past that. The FE model of the fixture included the accelerometer and triax block weight and geometry to generate mode shapes that would accurately simulate what was being gathered in a test.

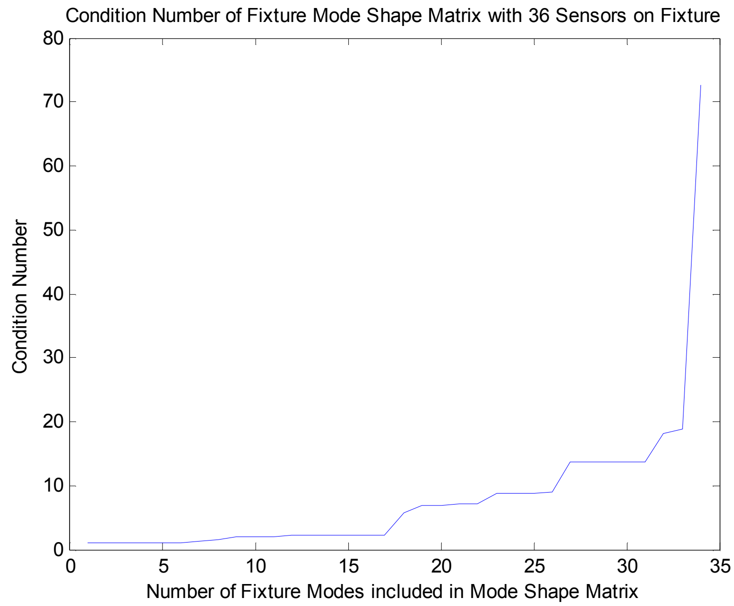

Figure 6 - Condition number of fixture mode shape matrix for 36 accelerometers

CAUSES OF ILL CONDITIONING FOR ADMITTANCE MODELING

There are some differences between the admittance and CMS substructure methods that cause different types of ill conditioning, so they will be addressed separately. First consider the admittance modeling. In essence, the connection to connection FRF matrices for each substructure have to be inverted to get the connection to connection FRF matrices for the full system. The subsystem FRFs are built up from the modal parameters of each subsystem. This means that the maximum rank of the subsystem FRF matrix is the number of modes of that subsystem. If the number of connection dof exceeds the number of modes for an FRF matrix, and that matrix must be inverted, the inversion will fail. In this application, there were 26 modes of the experimental substructure, so attempts to use 26 or more modal connections caused the inversions to be inaccurate or fail completely. The inverse of the fixture mode shape matrix is utilized to reduce the measurement connection FRFs down to the size of the number of fixture modes. As can be seen in Figure 6, if the number of modal connection dof gets large enough, this matrix will have inversion problems. The condition number of the matrices that were inverted was recorded during the admittance modeling process. The condition numbers increase with the number of modal connection dof chosen. When the number of dof gets close to the number of experimental modes, conditions numbers at some frequency lines hit 1e6, which caused spikes or dips at the associated frequency lines.

CAUSES OF ILL CONDITIONING FOR CMS

In the CMS formulation utilizing the fixture, the stiffness and mass of the two fixtures is actually subtracted based on a truncated modal estimate of the mass and stiffness. This estimate is not perfect, and therefore the final result can have complex eigenvalues or non-positive definite mass and stiffness matrices. The final mode shapes and frequencies are transformed back to physical coordinates to build the final FRFs. For this application, the complex mode shapes and eigenvalues seemed to have a negligible effect on the final FRFs. That is, whether the complex eigenvalues and associated eigenvectors were thrown away or utilized, did not affect the FRFs. The number of modal constraints is equal to the twice the number of fixture modal dof. Therefore the final number of dof =Na +Nb +Nf -2Nf= Na+Nb-Nf (where N = number of modes, and a, b and f denote subsystem a, b and the fixture). The FE model of the fixture allowed us to use as many modes as desired for Nf. It was found that the problem went unstable when Nf reached 36 or larger. This is the number of connection measurement accelerometers on the experimental substructure, so one would expect that the modal dof can not be defined uniquely if Nf > the number of measurement accelerometers.

Another important feature of this method is the approach to choosing the dependent modal dof for the constraint equations. Although the number of dof to be constrained is known, the best independent and dependent dof to choose when coupling the substructures is not. If the set of constrained dof cannot be determined uniquely from the unconstrained dof (the partition of the constraint matrix associated with the constrained dof is not invertible[2][3]), then the substructuring process may give erroneous results. Allen & Mayes[2] presented a method dubbed the Maximum Rank Coordinate Choice that circumvents this difficulty.

RESULTS FOR EXPERIMENTAL DRIVING POINT

The end of the beam was impacted in the axial direction on one corner and the driving point FRF acquired. In Figure 7 is the comparison between measured FRFs for the full system and the full system FE model FRFs calculated with 100 modes. The point of this figure is to illustrate that the FE model that was developed for this work was quite accurate. For the MCFS admittance model, twenty-six modes of the experimental substructure were utilized from extractions out to 4000 Hz. One hundred modes of the FE substructure were utilized. Eighteen modes of the fixture were utilized for the connection dof. For CMS, this results in a dynamic matrix with 100+26-18=108 dof. In admittance modeling, the important parameter is the size of the FRF matrix at the connection dof, which is a square 18x18 matrix for this case. The result that will be given for both admittance and CMS is synthesis of the driving point frequency response functions of the combined system at two points:

- 1000y which is an axial driving point at the tip of the beam on the experimental substructure
- 401x which is a lateral driving point on the side of the cylinder a few inches from the base

In Figure 8, the measured 1000y driving point FRF for the combined system is compared to the FRF predicted by admittance using the MCFS procedure. The CMS result (not shown) was very similar to the admittance result. A FE model of the experimental hardware, the plate and beam, was developed to perform virtual experiments. In Figure 9 the test data (blue) is compared with an admittance result (red) using 21 modes of the virtual experiment (out to 4000 Hz) and 100 modes of the FE model with the 48 physical connection dof and *no fixture*. The low frequency is certainly much better with the MCFS approach, indicating that it captures the flexibility much better than just free modes extracted to the same frequency. Of course, actually measuring the 48 physical connection dof motions might not be feasible in a real experiment.

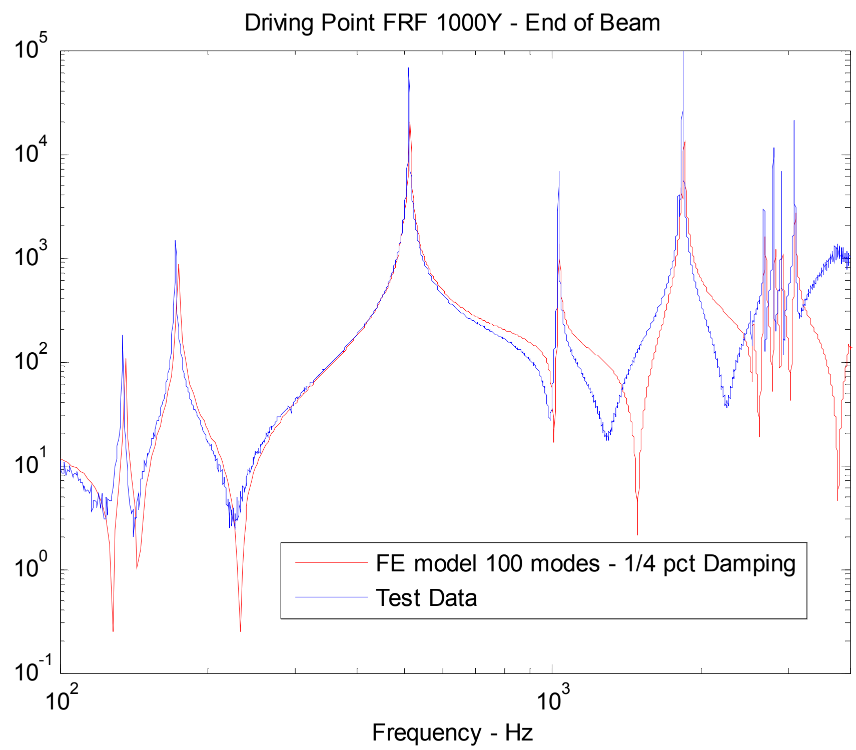

Figure 7 - 1000Y driving point full system measurement vs. full system fe model truncated to 100 modes

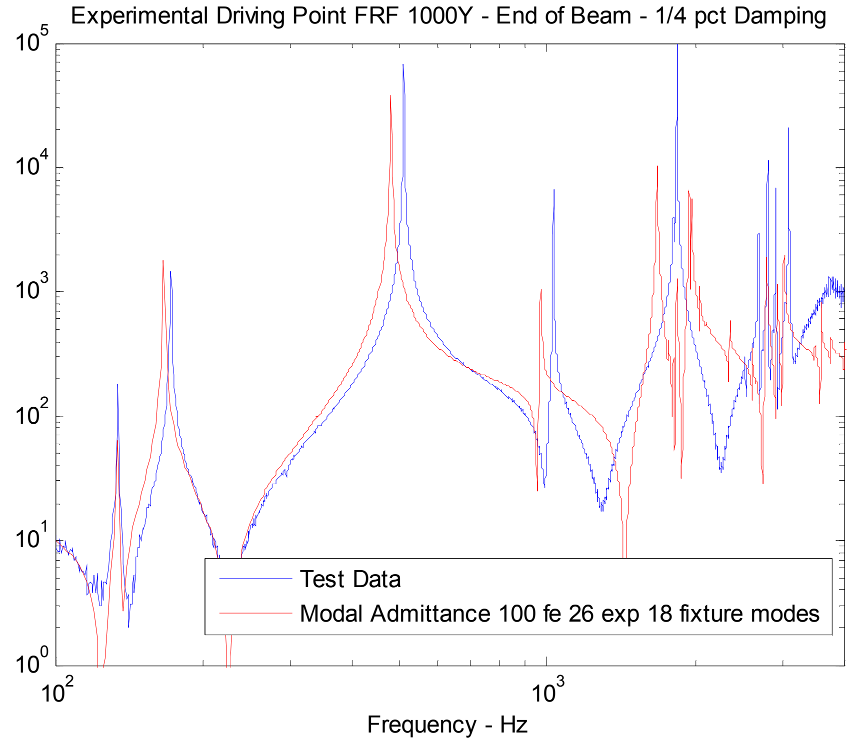

Figure 8 – Comparison of measured 1000y driving point and admittance prediction using MCFS method with 18 modal constraints

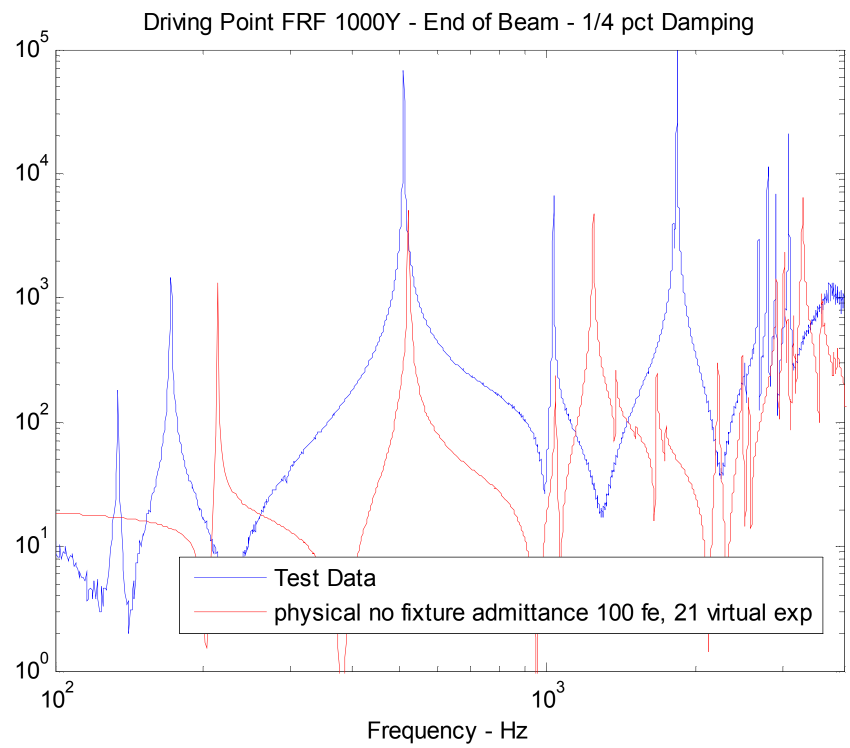

Figure 9 - Comparison of measured 1000y driving point and traditional admittance prediction coupling a 21 mode model of the virtual experiment substructure to a 100 mode model of the cylinder using 48 physical connection dof constraints but no fixtures.

RESULTS FOR FE DRIVING POINT

A driving point on the side of the cylinder in the radial direction was measured and is compared with the 100 mode result from the full FE model in Figure 10. The comparison is almost as good as the driving point on the end of the beam, again emphasizing the validity of the FE model used to aid this work. The test data is compared with the MCFS admittance prediction in Figure 11. For this FRF most of the resonances are dependent on the ovaling modes of the cylinder. Thus, the accuracy of this FRF is more dependent on the accuracy of the FE model of the cylinder than on the experimental substructure results. The lowest mode is dependent on the experimental result because it is the first bending mode, and this frequency is estimated fairly well. In Figure 12 is shown the comparison of a physical connection dof admittance model using 21 modes of the virtual experiment (out to 4000 Hz). Although the results look better than for the beam driving point in Figure 9, they are not as good as the MCFS admittance in which experimental modes were extracted out to 4000 Hz.

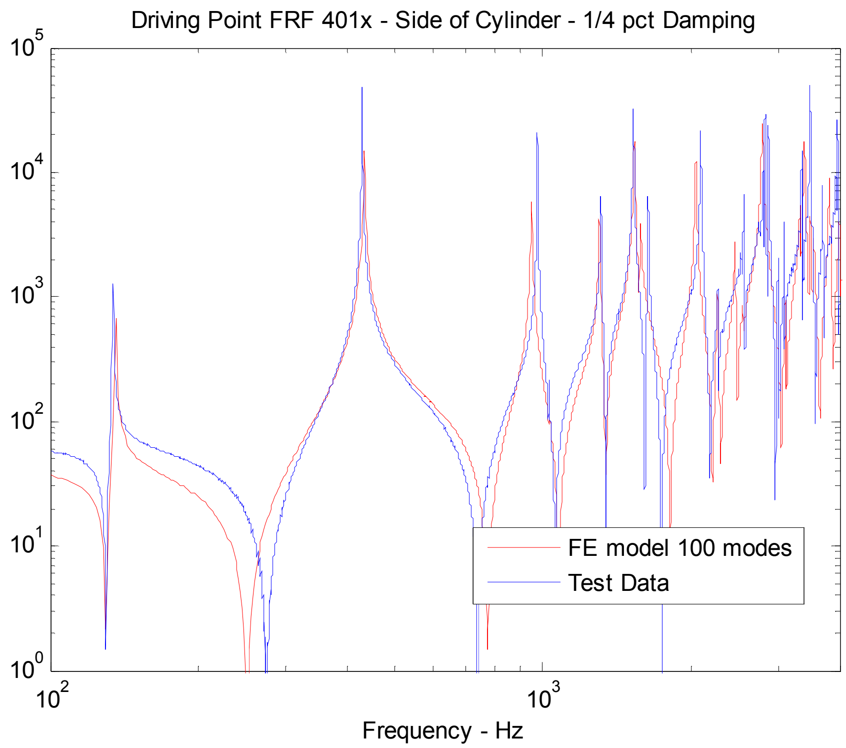

Figure 10 - Radial driving point on cylinder - full system measurement vs full system fe mode truncated to 100 Modes

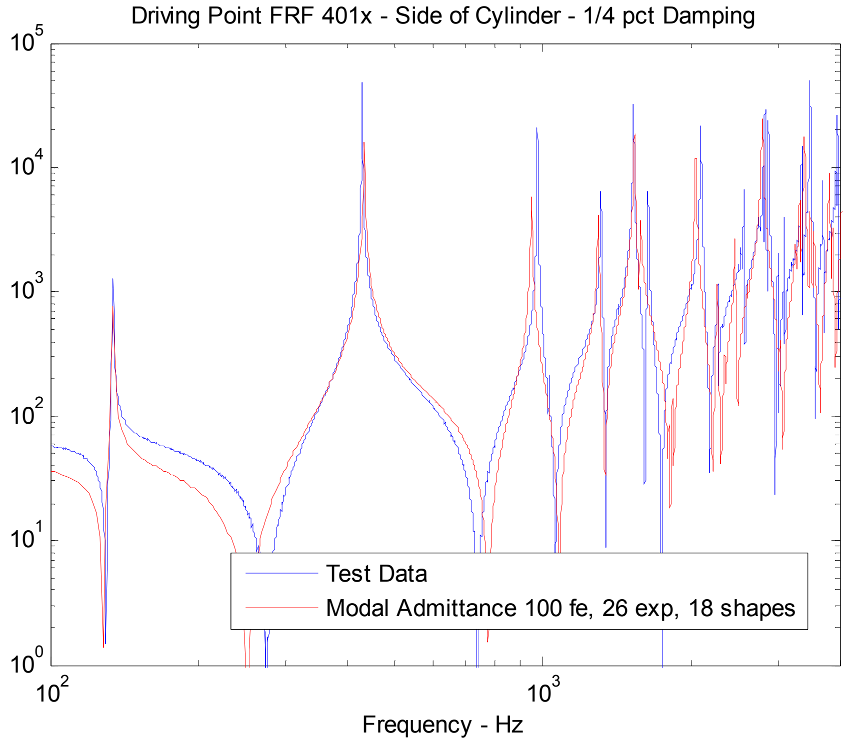

Figure 11 - Experimental full system measurement vs MCFS admittance result for driving point on side of cylinder

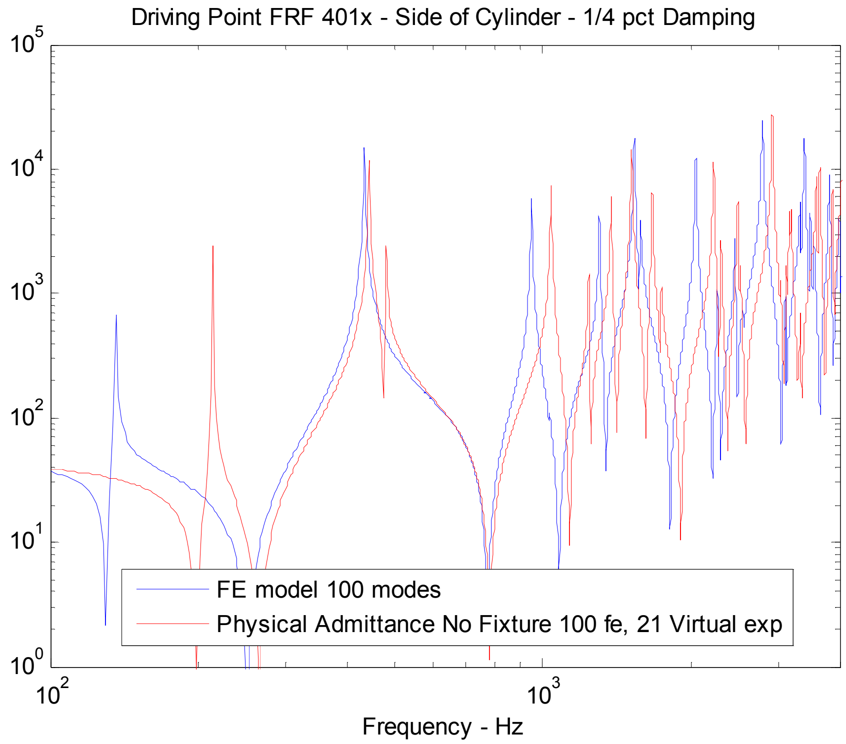

Figure 12- Comparing full system fe model to physical connection admittance prediction using 21 virtual experiment modes, 100 modes of analytical substructure and no fixture

ADVANTAGES OF MCFS APPROACH

The advantages of this approach are:
1. Rotation responses do not have to be measured;
2. Responses of the actual connection points do not have to be measured;
3. Forces at the connection points do not have to be measured;
4. The physical connection dof are reduced to a smaller number of modal connection dof;
5. Many fewer experimental modes are required for accurate response than with free modes of an experimental structure with no fixture;
6. Residual flexibility does not have to be measured if the fixture is designed properly, such that it captures the important flexibility of the joint in the mode shapes within the testable bandwidth;
7. Only one experimental fixture is required to capture response for all connection dof at multiple connection points;
8. The joint flexibility is captured in the experimental substructure;
9. The experimental substructure is automatically test verified (as opposed to a FE model) since it is derived from test data.

LIMITATIONS OF MCFS APPROACH

Limitations of this approach are:
1. The connection motion must be adequately captured as a linear sum of the fixture mode shapes;
2. It depends upon the accurate extraction of modal parameters from the modal test;
3. The frequency band is limited by the frequency band from which modes may be extracted;
4. The accuracy of the FE substructure also limits the accuracy of the system prediction;
5. Metrics for the accuracy of the final solution based on FE, experiment and fixture design have not been established.

FUTURE WORK

There remains considerable research to be done that could improve results and establish metrics that could be used to quantify uncertainties of the results. One area for work is the fixture design and mode shapes. The authors developed a fixture for which they thought the first 16 modes (up to about 1400 Hz) would be adequate to span the space of the motion. The shapes were inspected to see if a dishing mode of the ring was included to capture the first axial mode and bending modes of the ring captured bending motion that would be induced in the full system. The accelerometer set was optimized to capture these 16 modes. Slight improvements could be seen by utilizing more modes, but then the results got worse as the condition number of the mode shape matrix increased. The tradeoffs between mass, stiffness, number of modes in the band, number of modes captured by the accelerometer locations needs to be investigated. In future work, the authors plan to try stiffer and more massive fixtures, with fewer modes in the testable bandwidth.

The effect of uncertainty and errors in the FE models should also be considered, and methods sought to improve the robustness of the procedure. It was observed that the CMS procedure can result in a combined system that has non physical modes (complex natural frequencies or non-positive definite mass and/or stiffness). This did not seem to affect the accuracy of the predictions for this system, yet it should be investigated to see if these modes can be filtered out or rectified in the response. Similar results were found in this work whether the admittance or CMS procedure was used. The advantages and disadvantages of these two approaches should be explored more fully.

CONCLUSIONS

The Modal Constraint for Fixture and Subsystem (MCFS) approach in both CMS and admittance formulations has been utilized for the first time to connect experimental and analytical substructures with multiple connection points. The approach is significantly more robust than simply using free modes of a structure without a fixture. MCFS eliminates the classical experimental necessities of measuring rotations, using more than one fixture, and measuring residual flexibilities. Physical connection dof motions are not required, but the measurement dof on the fixture must adequately describe the fixture mode shapes, and the mode shapes of the fixture must adequately span the space of the connection motion. The number of measurement dof utilized for making the connection between substructures is reduced to the number of modal dof retained for the fixture. Parameters on fixture design and number of modes required are still open questions.

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APPENDIX A - ADMITTANCE THEORY

In this work, the admittance problem is solved twice. First the experimental substructure with fixture is connected to the analytical substructure with fixture. Then the full system with two fixtures is a substructure connected with a substructure of two negative fixtures. The algebra causes the FRFs of a substructure of two negative fixtures to be -0.5* the FRFs of a single fixture.

Two cases will be covered here. One is the case of input on substructure A and output on substructure B. The second is the case of input on substructure A and output on substructure A. System T is the result of combining the two substructures. Subscript r represents response dof. Subscript i represents force input dof. Subscript c represents connection dof

Case 1: Input on substructure A and output on substructure B

$$HT_{ri} = HB_{rc}\left(HB_{cc} + HA_{cc}\right)^{-1} HA_{ci} \quad \text{(1B)}$$

$$HT_{cc} = \left(HB_{cc}^{-1} + HA_{cc}^{-1}\right)^{-1} \quad \text{(2B)}$$

$$HT_{rc} = HB_{rc} - HB_{rc}HA_{cc}^{-1}\left(HB_{cc}^{-1} + HA_{cc}^{-1}\right)^{-1} \quad \text{(3B)}$$

$$HT_{ci} = HB_{cc}\left(HB_{cc} + HA_{cc}\right)^{-1} HA_{ci} \quad \text{(4B)}$$

The response and input degrees of freedom are physical. However, the MCFS approach converts the connection degrees of freedom to generalized dof of the fixture, reducing the size of the FRF matrices. The FRFs of each system A and B are first measured or calculated with physical dof. The physical connection dof, labeled with subscript p, (which come from measurement dof on the fixture of both subsystems) are converted to generalized connection dof in the following way where Φ_f is the mode shape matrix of the fixture at the measured dof.

$$HB_{rc} = HB_{rp}\Phi_f^{T+} \quad \text{(5B)}$$

$$HB_{cc} = \Phi_f^{+} HB_{pp}\Phi_f^{T+} \quad \text{(6B)}$$

$$HA_{cc} = \Phi_f^{+} HA_{pp}\Phi_f^{T+} \quad \text{(7B)}$$

$$HA_{ci} = \Phi_f^{+} HA_{pi} \quad \text{(8B)}$$

Case 2: Input and output dof on substructure A

$$HT_{ri} = HA_{ri} - HA_{rc}\left(HB_{cc} + HA_{cc}\right)^{-1} HA_{ci} \quad \text{(9B)}$$

The only matrix that was not given earlier is given below.

$$HA_{rc} = HA_{rp}\Phi_f^{T+} \quad \text{(10B)}$$